Une application du contrôle optimal au contrôle quantique

Journée de présentation des activités des étudiants

O. Cots

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Main Collaborations

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Two-level quantum control

Lindblad equation. Consider the bilinear dynamics in the two-level case,

$$\begin{aligned}\dot{x} &= -\Gamma x + u_2 z,\\ \dot{y} &= -\Gamma y - u_1 z,\\ \dot{z} &= \widetilde{\gamma} - \gamma z + u_1 y - u_2 x,\end{aligned}$$

where $2\Gamma \geq \gamma \geq |\widetilde{\gamma}|$ are dissipation parameters modelling the interaction with the environment (*e.g.*, molecular collisions).
The state $q = (x, y, z) \in \mathbf{R}^3$ belongs to the *Bloch ball* $\|q\| \leq 1$.
The control $u = (u_1, u_2)$ can be an electric or a magnetic field.

Energy minimization. The energy transfer between the system and the control field is minimized (fixed final time t_f),

$$\int_0^{t_f} (u_1^2 + u_2^2)\mathrm{d}t \to \min .$$

Applications

Spin-1/2 dynamics.

Control with a magnetic field spin-1/2 molecules in liquid phase. Exact model for applications in spectroscopy or medical imaging using Nuclear Magnetic Resonance.

Normal Hamiltonian

Spherical coordinates. Use $\rho > 0$ and (θ, φ) on the 2-sphere (universal covering of $\mathbf{S}^2 - \{N, S\}$),

$$q = \rho(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).$$

Maximized Hamiltonian. According to Pontryagin maximum principle, normal trajectories are projections of integral curves of

$$H(\rho,\theta,\varphi,p_\rho,p_\theta,p_\varphi) = H_0 + \frac{1}{2}(H_1^2 + H_2^2)$$

with

$$H_0 = -(\delta\sin^2\varphi + \gamma)\rho p_\rho - \delta\cos\varphi\sin\varphi\, p_\varphi - \frac{\widetilde{\gamma}}{\rho}(\rho p_\rho\cos\varphi - p_\varphi\sin\varphi),$$

$$H_1 = -\frac{p_\theta}{\tan\varphi}, \quad H_2 = p_\varphi,$$

and $\delta = \Gamma - \gamma$.

Integrable submodel

Liouville integrability. Set $r = \ln \rho$,

$$H_0 = -(\delta \sin^2 \varphi + \gamma) p_r - \delta \cos \varphi \sin \varphi \, p_\varphi - \widetilde{\gamma} e^{-r} (p_r \cos \varphi - p_\varphi \sin \varphi).$$

Proposition. For $\widetilde{\gamma} = 0$, coordinates r and θ (symmetry of revolution) are cyclic. The Hamiltonian flow is Liouville integrable with two linear first integrals, p_r and p_θ (Clairaut constant).

Reduction on the two-sphere. For $\widetilde{\gamma} = 0$, treat p_r as an additional parameter

$$H = -(\delta \sin^2 \varphi + \gamma) p_r - \delta \cos \varphi \sin \varphi \, p_\varphi + \frac{1}{2} \left(\frac{{p_\theta}^2}{\tan^2 \varphi} + {p_\varphi}^2 \right),$$

and use the quadrature

$$\dot{r} = -\delta \sin^2 \varphi + \gamma.$$

Integrable submodel

Mechanical system. On the level set $H = h$,

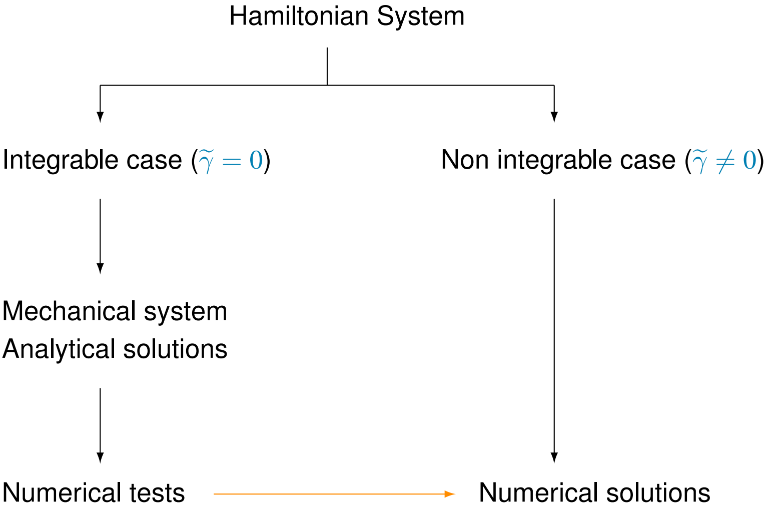
$$\frac{1}{2}\dot{\varphi}^2 + V(\varphi, p_r, p_\theta, \delta, \gamma) = h$$

where the potential is

$$V(\varphi, p_r, p_\theta, \delta, \gamma) = \frac{1}{2}\frac{{p_\theta}^2}{\tan^2\varphi} - \frac{\delta^2}{2}\cos^2\varphi\sin^2\varphi - (\delta\sin^2\varphi + \gamma)p_r.$$

Analytical solutions. In this case ($\widetilde{\gamma} = 0$) the normal extremals can be found explicitly in terms of Jacobi Elliptic functions or in terms of Weierstrass parameterization.

Sum up


Hamiltonian System
Integrable case ($\widetilde{\gamma} = 0$)
Non integrable case ($\widetilde{\gamma} \neq 0$)
Mechanical system
Analytical solutions
Numerical tests
Numerical solutions

The **hampath** code

- **hampath** package permits to solve and study smooth optimal control problem via indirect method ;
- Evolution of the `CotCot` code developed by J-B. Caillau ;
- *Fortran*90 kernel (Minpack, Lapack and Blas libraries and Tapenade) ;
- *Matlab* interface (ssolve, hampath, expdhvfun ...).

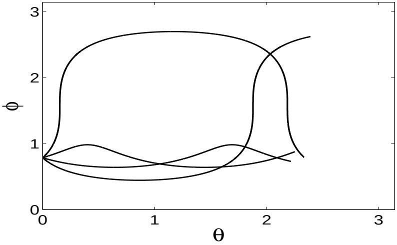
Matlab functions

ssolve uses HYBRJ (Minpack library) as NLE solver ($S(z) = 0$). Used to find solutions of optimal control problem ;

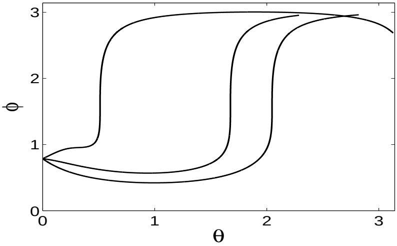
hampath uses DOPRI55 (RK(4)5 integrator) for the continuation. Permits to find solutions of difficult problems by homotopy from simpler ones ;

expdhvfun integrates the variational equations, with any initial condition. It is used to check the second order optimality conditions.

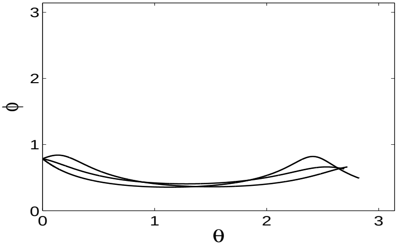
Extremals : Γ, γ_+, φ, p_ρ and p_θ fixed. $p_\varphi(0) = -1$, 0, 1 and 2.



$\widetilde{\gamma} = 0$



$\widetilde{\gamma} = -0.1 < 0$



$\widetilde{\gamma} = 0.1 > 0$

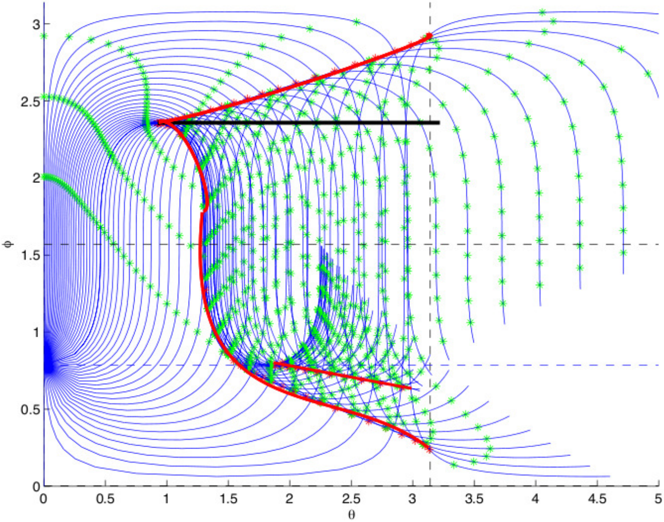
Cut and conjugate points

Cut point.
First point along an extremal s.t. it ceases to be globally minimizing.

Conjugate point.
First point along an extremal s.t. it ceases to be locally minimizing.

Conjugate points and isocost lines

Numerical computation. **hampath** code, $\widetilde{\gamma} = 0$, $\delta = 0$, $p_r = 0$ (no γ), $t_f = 11/2$, $\varphi_0 = \pi/4$, $h \leq 1/2$.



In prospect

- Improvement of **hampath** to treat multi-hamiltonian systems.
- Study quantum systems where two dissipative spin $1/2$ particles are controlled by the same magnetic field.