Optimal control model and resolution approaches for aircraft conflict avoidance problems via speed regulation

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1 Context
2 Optimal control model
3 Solution approaches
4 Application of the PMP
5 Conclusion
1 Context

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5 Conclusion
Aircraft conflict avoidance: separation standard

Two aircraft \((i\ \text{and} \ j)\) are said \textit{in conflict}\ if

\[
\begin{align*}
\| x_i - x_j \| &< 5 \text{ NM (}\approx 10 \text{ km)} \\
\| h_i - h_j \| &< 1000 \text{ ft (}\approx 300 \text{ m)}
\end{align*}
\]
Aircraft conflict avoidance: motivations

- *Permanent augmentation* of aircraft traffic.

- Separation still performed *manually* by the *air traffic controllers*.

  \[\Rightarrow\] Tools required for *decision support*.

  (SESAR project & ERASMUS project)
Aircraft conflict avoidance: motivations

- Permanent *augmentation* of aircraft traffic.

- Separation still performed *manually* by the *air traffic controllers*.

  \[\Rightarrow\] Tools required for *decision support*.  
  (SESAR project & ERASMUS project)

- Classical strategies for aircraft separation
Assumptions for the study:

- **tactical** phase (less than 20’ before conflict);
- **en-route** flight (cruise phase);
- **same altitude** for aircraft;
- **no heading changes**;
- **subliminal** velocity changes  
  \(\Rightarrow\) ERASMUS project.
Aircraft conflict avoidance: velocity regulation

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- Pathological case: The face-to-face.
Aircraft conflict avoidance: velocity regulation

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    (⇒ ERASMUS project).

- Pathological case: The face-to-face.

- Problem solving rhymes with Modeling...
1 Context

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5 Conclusion
Optimal control model ($\mathcal{P}$)
Optimal control model (\(\mathcal{P}\))

\[
\min_{\mathbf{u}} \sum_{i=1}^{n} \int_{t_0}^{t_f} u_i^2(t) dt
\]
Optimal control model (P)

\[
\min_u \sum_{i=1}^{n} \int_{t_0}^{t_f} u_i^2(t) dt
\]

\[\dot{v}_i(t) = u_i(t)\quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}\]

\[\dot{x}_i(t) = v_i(t)d_i\quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}\]
Optimal control model ($\mathcal{P}$)

$$\min_{u} \sum_{i=1}^{n} \int_{t_0}^{t_f} u_i^2(t) dt$$

$$\dot{v}_i(t) = u_i(t) \quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}$$

$$\dot{x}_i(t) = v_i(t)d_i \quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}$$

$$u_i(t)$$

$$\text{D}^2 - \|x_i(t) - x_j(t)\|^2 \leq 0 \quad \forall t \in [t_0, t_f] \quad \forall (i, j) \in \{1, \ldots, n\} \quad i < j$$
Optimal control model (\(\mathcal{P}\))

\[
\min_u \sum_{i=1}^{n} \int_{t_0}^{t_f} u_i^2(t) dt
\]

\[
\dot{v}_i(t) = u_i(t)
\]

\[
\dot{x}_i(t) = v_i(t) d_i
\]

\[
u_i \leq u_i(t) \leq u_i
\]

\[
\forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}
\]

\[
\text{separation} \quad \forall (i, j) \in \{1, \ldots, n\} \quad i < j
\]
Optimal control model (P)
Optimal control model (P)

\[
\min_u \sum_{i=1}^{n} \int_{t_0}^{t_f} u_i^2(t) dt
\]

\[
\dot{v}_i(t) = u_i(t) \quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}
\]

\[
\dot{x}_i(t) = v_i(t)d_i \quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}
\]

\[
u_i \leq u_i(t) \leq \overline{u}_i \quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}
\]

\[
v_i \leq v_i(t) \leq \overline{v}_i \quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}
\]
Optimal control model (\(\mathcal{P}\))

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\min_u \sum_{i=1}^{n} \int_{t_0}^{t_f} u_i^2(t) dt
\]

\[
\dot{v}_i(t) = u_i(t) \quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}
\]

\[
\dot{x}_i(t) = v_i(t) d_i \quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}
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\[
u_i \leq u_i(t) \leq \bar{u}_i \quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}
\]

\[
v_i \leq v_i(t) \leq \bar{v}_i \quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}
\]

\[
x_i(t_0) = x_i^0 \quad v_i(t_0) = v_i^0 \quad \forall i \in \{1, \ldots, n\}
\]
Optimal control model (\(\mathcal{P}\))

\[
\min_u \quad \sum_{i=1}^{n} \int_{t_0}^{t_f} u_i^2(t)\,dt
\]

\[
\begin{align*}
\dot{v}_i(t) &= u_i(t) \\
\dot{x}_i(t) &= v_i(t)\,d_i \\
u_i &\leq u_i(t) \leq \overline{u}_i \\
v_i &\leq v_i(t) \leq \overline{v}_i \\
x_i(t_0) &= x_i^0 \quad v_i(t_0) = v_i^0 \\
x_i(t_f) &= x_i^f \quad v_i(t_f) = v_i^f
\end{align*}
\]
Optimal control model (\(\mathcal{P}\))

\[
\begin{align*}
\min_u & \quad \sum_{i=1}^{n} \int_{t_0}^{t_f} u_i^2(t) dt \\
\dot{v}_i(t) &= u_i(t) \\
\dot{x}_i(t) &= v_i(t) d_i \\
u_i &\leq u_i(t) \leq \bar{u}_i \\
v_i &\leq v_i(t) \leq \bar{v}_i \\
x_i(t_0) &= x_i^0 \\
v_i(t_0) &= v_i^0 \\
x_i(t_f) &= x_i^f \\
v_i(t_f) &= v_i^f \\
D^2 - \| x_i(t) - x_j(t) \|^2 &\leq 0 \\
\forall t \in [t_0, t_f] &\quad \forall i \in \{1, \ldots, n\} \\
\forall t \in [t_0, t_f] &\quad \forall i \in \{1, \ldots, n\} \\
\forall t \in [t_0, t_f] &\quad \forall i \in \{1, \ldots, n\} \\
\forall t \in [t_0, t_f] &\quad \forall i \in \{1, \ldots, n\} \\
\forall t \in [t_0, t_f] &\quad \forall (i, j) \in \{1, \ldots, n\}^2 \text{ and } i < j
\end{align*}
\]
For each instant $t \in [t_0, t_f]$, for each aircraft $i$: 

Acceleration as command
Acceleration as command

For each instant \( t \in [t_0, t_f] \), for each aircraft \( i \):

\[
\dot{v}_i(t) = u_i(t)
\]

The acceleration \( u_i(t) \)
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Acceleration as command

For each instant \( t \in [t_0, t_f] \), for each aircraft \( i \):

\[
\dot{v}_i(t) = u_i(t)
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Acceleration as command

For each instant \( t \in [t_0, t_f] \), for each aircraft \( i \):

The acceleration \( u_i(t) \)

\[ \dot{v}_i(t) = u_i(t) \]
Acceleration as command

For each instant $t \in [t_0, t_f]$, for each aircraft $i$:

The acceleration $u_i(t)$

$$\dot{v}_i(t) = u_i(t)$$

$\downarrow$

The velocity $v_i(t)$
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Acceleration as command

For each instant \( t \in [t_0, t_f] \), for each aircraft \( i \):

The acceleration \( u_i(t) \)

\[
\dot{v}_i(t) = u_i(t)
\]

\[
\downarrow
\]

The velocity \( v_i(t) \)

\[
\dot{x}_i(t) = v_i(t)d_i
\]
For each instant $t \in [t_0, t_f]$, for each aircraft $i$:

**The acceleration $u_i(t)$**

$$\dot{v}_i(t) = u_i(t)$$

**The velocity $v_i(t)$**

$$\dot{x}_i(t) = v_i(t)d_i$$
Acceleration as command

For each instant $t \in [t_0, t_f]$, for each aircraft $i$:

The acceleration $u_i(t)$

$$\dot{v}_i(t) = u_i(t)$$

⇒

The velocity $v_i(t)$

$$\dot{x}_i(t) = v_i(t)d_i$$

⇒

The position $x_i(t)$
Acceleration as command

For each instant \( t \in [t_0, t_f] \), for each aircraft \( i \):

The acceleration \( u_i(t) \)

\[
\dot{v}_i(t) = u_i(t) \\
\downarrow
\]

The velocity \( v_i(t) \)

\[
\dot{x}_i(t) = v_i(t) d_i \\
\downarrow
\]

The position \( x_i(t) \)
Optimal control methods

Direct methods:

- Simple implementation without \textit{a priori} knowledge
- Low sensitivity on the choice of the initial condition
- Easy consideration of state constraints
- Low or medium numerical precision
- Efficient in small dimension

Indirect methods:

- \textit{A priori} knowledge required on the solution structure
- High sensitivity on the choice of the initial condition
- Theoretical difficulties to consider state constraints
- Very high numerical precision
- Efficient in any dimension
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Direct method: time discretization

The time discretization step: $h = \frac{t_f - t_0}{N}$

\[
\forall k \in \{0, \ldots, N - 1\} \quad \forall k \in \{0, \ldots, N\} \quad \forall k \in \{0, \ldots, N\}
\]

\[
\forall i \in \{0, \ldots, n\} \quad u_i^{(k)} \sim u_i(t_k) \quad v_i^{(k)} \sim v_i(t_k) \quad x_i^{(k)} \sim x_i(t_k)
\]

\[
\begin{align*}
&u_i^{(0)} + u_i^{(1)} + u_i^{(2)} \\
&= t_0 + kh = t_f
\end{align*}
\]
Direct method: time discretization (NLP)
Direct method: time discretization (NLP)

$$\min_u \sum_{i=1}^{n} \sum_{k=0}^{N-1} (u_i^{(k)})^2$$
Direct method: time discretization (NLP)

\[
\min_u \sum_{i=1}^{n} \sum_{k=0}^{N-1} (u_i^{(k)})^2
\]

\[v_i^{(k+1)} = NUM_{v_i}(u_i^{(k)}) \quad \forall k \in \{0, \ldots, N-1\} \ \forall i \in \{1, \ldots, n\}\]

\[x_i^{(k+1)} = NUM_{x_i}(v_i^{(k)}) \quad \forall k \in \{0, \ldots, N-1\} \ \forall i \in \{1, \ldots, n\}\]
Direct method: time discretization (NLP)

\[ \min_u \sum_{i=1}^{n} \sum_{k=0}^{N-1} (u_i^{(k)})^2 \]

\[ v_i^{(k+1)} = NUM v_i(u_i^{(k)}) \quad \forall k \in \{0, \ldots, N-1\} \ \forall i \in \{1, \ldots, n\} \]

\[ x_i^{(k+1)} = NUM x_i(v_i^{(k)}) \quad \forall k \in \{0, \ldots, N-1\} \ \forall i \in \{1, \ldots, n\} \]

\[ u_i \leq u_i^{(k)} \leq u_i \quad \forall k \in \{0, \ldots, N-1\} \ \forall i \in \{1, \ldots, n\} \]

\[ v_i \leq v_i^{(k)} \leq v_i \quad \forall k \in \{0, \ldots, N\} \ \forall i \in \{1, \ldots, n\} \]
Direct method: time discretization (NLP)

\[ \min_u \sum_{i=1}^{n} \sum_{k=0}^{N-1} (u_i^{(k)})^2 \]

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\[ x_i^{(k+1)} = NUM x_i(v_i^{(k)}) \quad \forall k \in \{0, \ldots, N-1\} \quad \forall i \in \{1, \ldots, n\} \]

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\[ v_i \leq v_i^{(k)} \leq \bar{v}_i \quad \forall k \in \{0, \ldots, N\} \quad \forall i \in \{1, \ldots, n\} \]

\[ x_i^{(0)} = x_i^0 \quad v_i^{(0)} = v_i^0 \quad \forall i \in \{1, \ldots, n\} \]

\[ x_i^{(N)} = x_i^f \quad v_i^{(N)} = v_i^f \quad \forall i \in \{1, \ldots, n\} \]
Direct method: time discretization (NLP)

\[
\min_u \sum_{i=1}^{n} \sum_{k=0}^{N-1} (u_i^{(k)})^2
\]

\[
v_i^{(k+1)} = NUM_v_i(u_i^{(k)}) \quad \forall k \in \{0, \ldots, N-1\} \forall i \in \{1, \ldots, n\}
\]

\[
x_i^{(k+1)} = NUM_x_i(v_i^{(k)}) \quad \forall k \in \{0, \ldots, N-1\} \forall i \in \{1, \ldots, n\}
\]

\[
u_i \leq u_i^{(k)} \leq \bar{u}_i \quad \forall k \in \{0, \ldots, N-1\} \forall i \in \{1, \ldots, n\}
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v_i \leq v_i^{(k)} \leq \bar{v}_i \quad \forall k \in \{0, \ldots, N\} \forall i \in \{1, \ldots, n\}
\]

\[
x_i^{(0)} = x_i^0 \quad v_i^{(0)} = v_i^0 \quad \forall i \in \{1, \ldots, n\}
\]

\[
x_i^{(N)} = x_i^f \quad v_i^{(N)} = v_i^f \quad \forall i \in \{1, \ldots, n\}
\]

\[
D^2 - \|x_i^{(k)} - x_j^{(k)}\|^2 \leq 0 \quad \forall k \in \{0, \ldots, N\} \quad \text{separation}
\]

\[
\forall (i, j) \in \{1, \ldots, n\}^2 \text{ and } i < j
\]
Complexity of NLP

- number of variables: $O(np)$
- number of constraints: $O(n^2p + p^2)$

with $n$ number of aircraft, $p$ number of time subdivisions.
Direct method: computational complexity

- **Complexity of NLP**
  - number of variables: $O(np)$
  - number of constraints: $O(n^2p + p^2)$

  with $n$ number of aircraft, $p$ number of time subdivisions.

- *Exempli gratia*: a 2-aircraft conflict avoidance problem with a time horizon: 30’, and a time subdivision: 15”.
  - more than 240 variables;
  - more than 9000 constraints.
IDEA: using two steps

\[(\text{detection}) \quad h = \frac{t_f - t_0}{N} \quad << \quad h' = \frac{t_f - t_0}{N'} \quad (\text{control})\]

\[\forall i \in \{0, \ldots, n\} \quad u_i^{(l)} \sim u_i(t'_l) \quad \forall k \in \{0, \ldots, N\} \quad v_i^{(k)} \sim v_i(t_k) \quad x_i^{(k)} \sim x_i(t_k)\]

\[\forall l \in \{0, \ldots, N' - 1\}\]

\[u_i^{(0)} + u_i^{(1)} + u_i^{(2)}\]

\[t'_0 \quad \overline{\text{Step}_\text{control}} \quad t'_l = t'_0 + lh'\]

\[t_0 \quad \overline{\text{Step}_\text{detection}} \quad t_k = t_0 + kh \quad t_N = t_f\]
Time discretization: illustration

Acceleration solution curves for 1' and 15" control time steps.

Speed solution curves for 1' and 15" control time steps.
Specificity of aircraft conflict avoidance

For each couple of aircraft, do we really need to check separation constraint everywhere in the considered air sector?

- separation constraint must be checked when aircraft get close to each other
- separation is useless in certain “regions”
IDEA

✓ Decompose the problem into subproblems corresponding to different “regions”

✓ Apply direct and indirect optimal control methods depending on the “regions”
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Problem decomposition

Space decomposition

aircraft $i$

aircraft $j$
Problem decomposition

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Space decomposition

aircraft $i$

5 NM

$x_{ij}^{enter}$

aircraft $j$
Problem decomposition

Space decomposition

aircraft $i$

5 NM

$x_{ij}$

$5 \text{ NM}$

$\text{Exit}$

aircraft $j$

$\text{Context}$

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aircraft \( i \)

5 NM

\( x_{\text{enter}}^{ji} \)

aircraft \( j \)
Problem decomposition

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aerofcraft $i$

aerofcraft $j$

$x_{exit}^{ji}$

5 NM
Problem decomposition

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Problem decomposition: definitions

Space decomposition

aircraft $i$

aircraft $j$

$ij$ enter

$ji$ enter

$ij$ exit

$ji$ exit

prezone

zone

postzone

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Time decomposition

aircraft $i$ ($v_i$)

$t_{1 \_max}^i$  $t_{2 \_max}^i$  

$t_1$  $t_2$

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Problem decomposition

Time decomposition

aircraft $i$ ($\bar{v}_i$)

$t_{1\text{-}min}^i$ $t_{2\text{-}min}^i$

$t_i$

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Problem decomposition

Time decomposition

\[ t_{1\text{-}min}^j \quad t_{2\text{-}min}^j \]

aircraft \( j \) \( (\bar{v}_j) \)
Problem decomposition

Time decomposition

aircraft $i$

$\text{t}^i_{1\_min}$ $\text{t}^i_{1\_max}$ $\text{t}^i_{2\_min}$ $\text{t}^i_{2\_max}$

aircraft $j$

$\text{t}^j_{1\_min}$ $\text{t}^j_{1\_max}$ $\text{t}^j_{2\_min}$ $\text{t}^j_{2\_max}$

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Time decomposition

aircraft $i$

$t^i_{1_{-min}}$

$t^i_{2_{-max}}$

aircraft $j$

$t^j_{1_{-min}}$

$t^j_{2_{-max}}$

time
Problem decomposition: definitions

Time decomposition

aircraft $i$

$\min t_1^i \quad \max t_2^i$

prezone \quad zone \quad postzone

$\min \ t_1 \quad \max \ t_2$

aircraft $j$

$t_1^j \quad t_2^j$

$t_1^i \quad t_2^i$
Combining optimal control approaches

**Indirect method**

- Prezone: Application of PMP
- Zone: Using time discretization with two steps
- Postzone: Application of PMP

**Direct method**

- Prezone: Analytical solution
- Zone: Using time discretization with two steps
- Postzone: Analytical solution
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Optimal control: original model ($\mathcal{P}$)

$$\min_u \sum_{i=1}^{n} \int_{t_0}^{t_f} u_i^2(t) dt$$

$$\dot{v}_i(t) = u_i(t) \quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}$$

$$\dot{x}_i(t) = v_i(t) d_i \quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}$$

$$u_i \leq u_i(t) \leq \bar{u}_i \quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}$$

$$v_i \leq v_i(t) \leq \bar{v}_i \quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}$$

$$x_i(t_0) = x_i^0 \quad v_i(t_0) = v_i^0 \quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}$$

$$x_i(t_f) = x_i^f \quad v_i(t_f) = v_i^f \quad \forall t \in [t_0, t_f] \quad \forall i \in \{1, \ldots, n\}$$

$$D^2 - \| x_i(t) - x_j(t) \|^2 \leq 0 \quad \forall t \in [t_0, t_f] \quad \text{separation}$$

$$\forall (i, j) \in \{1, \ldots, n\}^2 \text{ and } i < j$$
Optimal control on the postzone

Problem \( (P_i) \) for the aircraft \( i \)

\[
\min_{u_i} \int_{t_2}^{t_f} u_i^2(t) dt
\]

\[
\dot{v}_i(t) = u_i(t) \quad \forall t \in [t_2, t_f]
\]

\[
\dot{x}_i(t) = v_i(t) d_i \quad \forall t \in [t_2, t_f]
\]

\[
u_i \leq u_i(t) \leq \overline{u_i} \quad \forall t \in [t_2, t_f]
\]

\[
x_i(t_2) = x_i^{t_2} \quad v_i(t_2) = v_i^{t_2}
\]

\[
x_i(t_f) \text{ free} \quad v_i(t_f) = v_i^0
\]
Optimal control on the postzone

Application of the PMP on $(\mathcal{P}_i)$:

- Co-state variables $z^i_0, z^i_1, z^i_2, z^i_3$.
  (where $z^i_1, z^i_2, z^i_3$ associated to $x^X_i, x^Y_i$ and respectively $v_i$.)

- Hamiltonian $H_i$:

$$H_i = z^i_0 u^2_i + z^i_1 v_i d^X_i + z^i_2 v_i d^Y_i + z^i_3 u_i .$$ (1)

- Co-state equations:

$$\dot{z}^i_1 = - \frac{\partial H_i}{\partial x^X_i} = 0 , \quad \dot{z}^i_2 = - \frac{\partial H_i}{\partial x^Y_i} = 0 ,$$

$$\dot{z}^i_3 = - \frac{\partial H_i}{\partial v_i} = -(z^i_1 d^X_i + z^i_2 d^Y_i) .$$ (2)
Optimal control on the *postzone*

Solution for the aircraft *i*

\[
\begin{align*}
S_i & \quad \left\{ 
\begin{array}{l}
    u_i(t) &= \frac{v_i^{t_f} - v_i^{t_2}}{t_f - t_2} \quad (= \text{constant}!) , \\
    v_i(t) &= \frac{v_i^{t_f} - v_i^{t_2}}{t_f - t_2} (t - t_f) + v_i^{t_f} , \\
    x_i^X(t) &= \frac{v_i^{t_f} - v_i^{t_2}}{t_f - t_2} \cdot \frac{d_i^X}{2} t^2 + \left( v_i^{t_f} - \frac{v_i^{t_f} - v_i^{t_2}}{t_f - t_2} \right) \cdot \frac{d_i^X}{2} t \\
    & \quad \quad - \left( \frac{v_i^{t_f} - v_i^{t_2}}{t_f - t_2} \right) (t_2 - t_f) + v_i^{t_f} \right) \cdot d_i^X t_2 + x_i^{X t_2} , \\
    x_i^Y(t) &= \frac{v_i^{t_f} - v_i^{t_2}}{t_f - t_2} \cdot \frac{d_i^Y}{2} t^2 + \left( v_i^{t_f} - \frac{v_i^{t_f} - v_i^{t_2}}{t_f - t_2} \right) \cdot \frac{d_i^Y}{2} t \\
    & \quad \quad - \left( \frac{v_i^{t_f} - v_i^{t_2}}{t_f - t_2} \right) (t_2 - t_f) + v_i^{t_f} \right) \cdot d_i^Y t_2 + x_i^{Y t_2} .
\end{array}
\right\}
\end{align*}
\]
Problem \((P'_i)\) for the aircraft \(i\)

\[
\min_{u_i} \int_{t_0}^{t_1} u_i^2(t) dt
\]

\[
\dot{v}_i(t) = u_i(t) \quad \forall t \in [t_0, t_1]
\]

\[
\dot{x}_i(t) = v_i(t) d_i \quad \forall t \in [t_0, t_1]
\]

\[
u_i \leq u_i(t) \leq \bar{u}_i \quad \forall t \in [t_0, t_1]
\]

\[
\underline{v}_i \leq v_i(t) \leq \bar{v}_i \quad \forall t \in [t_0, t_1]
\]

\[
x_i(t_0) = x^0_i \quad v_i(t_0) = v^0_i
\]

\[
x_i(t_1) \text{ unknown} \quad v_i(t_1) \text{ unknown}
\]
Prezone & Postzone: models

Prezone model is more difficult than postzone model:

- terminal conditions (position & velocity) are unknown
- velocity constraint (state constraint) has to be checked

\[
\begin{align*}
\min_{u_i} & \int_{t_0}^{t_1} u_i^2(t) dt \\
\dot{v}_i(t) &= u_i(t) \\
x_i(t) &= v_i(t) d_i \\
u_i &\leq u_i(t) \leq \overline{u}_i \\
v_i &\leq v_i(t) \leq \overline{v}_i \\
x_i(t_0) &= x_i^0 \\
v_i(t_0) &= v_i^0 \\
x_i(t_1) &\text{ unknown} \\
v_i(t_1) &\text{ unknown}
\end{align*}
\]

\[
\begin{align*}
\min_{u_i} & \int_{t_2}^{t_f} u_i^2(t) dt \\
\dot{v}_i(t) &= u_i(t) \\
x_i(t) &= v_i(t) d_i \\
u_i &\leq u_i(t) \leq \overline{u}_i \\
v_i &\leq v_i(t) \leq \overline{v}_i \\
x_i(t_2) &= x_i^{t_2} \\
v_i(t_2) &= v_i^{t_2} \\
x_i(t_f) &\text{ free} \\
v_i(t_f) &= v_i^0
\end{align*}
\]

prezone model  \hspace{2cm} postzone model
Thanks to the *Pontryagin Maximum Principle*,
the optimal control (acceleration) is:

**Linear** while
velocity constraint is inactive,

**Null** otherwise,

on the *prezone* (i.e., \([t_0, t_1]\));

**Constant** on the *postzone* (i.e., \([t_2, t_f]\)).

Example of speed solution curves *without* and *with* velocity constraint
Optimal control model and resolution approaches for aircraft conflict avoidance problems via speed regulation

Loïc Cellier, Sonia Cafieri & Frédéric Messine

1. Context
2. Optimal control model
3. Solution approaches
4. Application of the PMP
5. Conclusion
Conclusion

- Extensive numerical tests.

- Combining optimal control and integer programming methods.

- From local optimality to global optimality: Investigation on the mathematical model properties.
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Thanks for your attention / questions?