Limited Memory Preconditioners for symmetric indefinite systems. Application to saddle point problems

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Sommaire

Problem to solve

Saddle point systems (fluid mechanics, optimisation...)

$$Ax = b \Longleftrightarrow \begin{pmatrix} K & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

$K \in \mathbb{R}^{n\times n}$ symmetric, $B \in \mathbb{R}^{m\times n}$, $m \leq n$ and
$C \in \mathbb{R}^{m\times m}$ symmetric positive semidefinite.

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Nonlinear problems in solid mechanics

- Dualizing Dirichlet boundary conditions with "double Lagrange" principle
- Newton-like method $\Longrightarrow$ sequence of linear systems to solve:

$$A_i x_i = b_i \text{ avec } A_i = \begin{pmatrix} K_i & B^T & B^T \\ B & -\alpha I & \alpha I \\ B & \alpha I & -\alpha I \end{pmatrix}, \quad \alpha \in \mathbb{R}^{+*}$$

Preconditioning [Benzi et al.,2005]

$$Ax = b \Longleftrightarrow \begin{pmatrix} K & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

Krylov subspace methods $\Rightarrow$ Preconditioning

Examples:

- $P_{c,d} = \begin{pmatrix} I & 0 \\ cBK_0^{-1} & I \end{pmatrix} \begin{pmatrix} K_0 & 0 \\ 0 & S_0 \end{pmatrix} \begin{pmatrix} I & dK_0B^T \\ 0 & I \end{pmatrix}$,
  $K_0 \simeq K$ et $S_0 \simeq -C - BK^{-1}B^T$

  $\Rightarrow$ block diagonal, block triangular, constraint
- Inexact constraint preconditioners ($B_0 \simeq B$)
- Multigrid, domain decomposition
- ...

Objective: A class of Limited Memory Preconditioners (LMP) is available in a symmetric positive definite framework.

The aim is to generalize this class to symmetric **indefinite** systems.

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Objective

- **Framework**: Solving a sequence of linear systems $A_i x_i = b_i$, with A_i SPD, using the Conjugate Gradient (CG) method.
- A first SPD preconditioner is available (*first-level*)
- It is possible to compute (cheaply) some directions slowing down the convergence of the CG method (e.g. Ritz, Harmonic Ritz...)
- **Goal**: Construct a *second-level* preconditioner with these directions to speed up the CG convergence.

Motivation

A^{-1} is the inverse Hessian of $q(x) = \frac{1}{2}x^T A x - b^T x$
The idea is to use the BFGS Hessian approximation:

$$H_i = (I_n - \frac{y_i s_i^T}{y_i^T s_i})^T H_{i-1} (I_n - \frac{y_i s_i^T}{y_i^T s_i}) + \frac{s_i s_i^T}{y_i^T s_i}$$

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LMP definition:

$$H = (I_n - S(S^T AS)^{-1} S^T A)M(I_n - AS(S^T AS)^{-1} S^T) + S(S^T AS)^{-1} S^T$$

with k linearly independent column vectors in S, and M SPD.

Properties

First property

- H is symmetric

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Spectral properties:

- $HAS = S$, namely the k vectors of S are eigenvectors of HA associated to 1.
- H is SPD and the spectrum of HA is contracted relatively to the MA one. [Gratton et al., Th. 3.4, 2011]

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Invariance property:

- Let $Z = SX$ where X is a nonsingular matrix of order k. Then H with Z instead of S remains unchanged. [Gratton et al., th. 3.1, 2011]

  $\Rightarrow$ Construction of an A-conjugate basis of $range(S)$ such as:

$$H = (I_n - ZZ^T A)M(I_n - AZZ^T) + ZZ^T$$

Arnoldi/Lanczos, general case

The Arnoldi process constructs an orthonormal basis of the Krylov subspace $\mathcal{K}_m = \mathcal{K}_m(A, r_0)$ (Lanczos if A sym.)

$$AV_m = V_{m+1}H_{m+1,m} = V_m H_m + h_{m+1,m} v_{m+1} e_m^T$$

$H_{m+1,m}$ is an Hessenberg matrix
H_m is tridiagonal in the symmetric case

Remark: when $m = n$, $A = V_n H_n V_n^T$, A and H_n have same spectra.

Ritz/Harmonic Ritz

We search an approximation (w_i, β_i) of an eigenpair such as

$$\begin{cases} w_i \in \mathcal{K}_m \\ Aw_i - \beta_i w_i \perp \mathcal{L}_m \end{cases}$$

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- **Ritz**: $\mathcal{L}_m = \mathcal{K}_m$, $(w_i, \beta_i) = (V_m y_i, \beta_i)$

  with (y_i, β_i) eigenpair of H_m

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- **Harmonic Ritz**: $\mathcal{L}_m = A\mathcal{K}_m$, $(w_i, \beta_i) = (V_m y_i, \beta_i)$

  with (y_i, β_i) eigenpair of $(H_m + (H_m^T)^{-1} e_m h_{m+1,m}^2 e_m^T)$

Sommaire

Notation

Formula:

$$H = (I_n - S(S^T AS)^{-1} S^T A) M (I_n - AS(S^T AS)^{-1} S^T) + S(S^T AS)^{-1} S^T$$

Notation:
It is possible to use $M^{\frac{1}{2}} A M^{\frac{1}{2}}$ as matrix and I_n as *first-level* preconditioner. Subsequently, we will refer to the LMP H as follows:

$$H = (I_n - S(S^T KS)^{-1} S^T K) H_0 (I_n - KS(S^T KS)^{-1} S^T) + S(S^T KS)^{-1} S^T$$

Sign matrix

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Idea: Decouple H to work separately in the positive (I^+) and negative (I^-) invariant subspaces of K.

How to: project a part of S in I^+ and the other in I^-. To do this, we will use the sign matrix of K, denoted X_K, and the projectors:

$$P_+ = \frac{I_n + X_K}{2} \quad \text{and} \quad P_- = \frac{I_n - X_K}{2}$$

Definition of the preconditioner

Notation

$$S = [S_+, S_-] \text{ with } S_+ \in I^+ \text{ et } S_- \in I^-$$

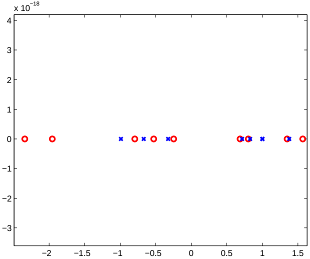
Decoupling

$$\begin{aligned} H = \; & (I_n - [S_+, S_-] \begin{pmatrix} S_+^T K S_+ & 0 \\ 0 & S_-^T K S_- \end{pmatrix}^{-1} [S_+, S_-]^T K) \times H_0 \times \\ & (I_n - K[S_+, S_-] \begin{pmatrix} S_+^T K S_+ & 0 \\ 0 & S_-^T K S_- \end{pmatrix}^{-1} [S_+, S_-]^T) + \\ & [S_+, S_-] \begin{pmatrix} S_+^T K S_+ & 0 \\ 0 & S_-^T K S_- \end{pmatrix}^{-1} [S_+, S_-]^T \end{aligned}$$

Theoretical results

Contraction on each side of the real axis

$\mathcal{P}_I$ property



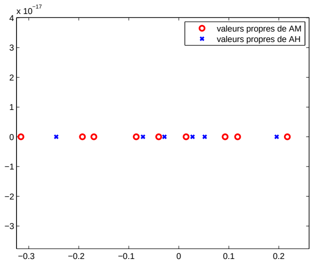
legend: $\sigma(KH_0)$ in red and $\sigma(KH)$ in blue

Assumptions: H_0 **symmetric positive definite** and $KH_0 = H_0K$

Theoretical results

Global contraction

$\mathcal{P}_g$ property



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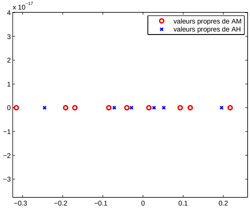
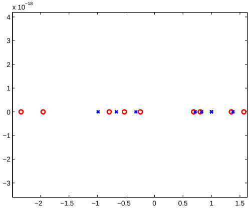
Assumptions: H_0 **symmetric indefinite** and $KH_0 = H_0 K$

Theoretical results

Spectral contraction

$\mathcal{P}_g$ property (orange ●) | $\mathcal{P}_l$ property (green ●)





	$K = A, H_0 = M$	$K = AM, H_0 = I$	$K = M^{\frac{1}{2}} A M^{\frac{1}{2}}, H_0 = I$
M SPD, $MA = AM$	green ●	green ●	green ●
M NPD, $MA = AM$	orange ●	green ●	×
M SPD, $MA \neq AM$	red ●	red ●	green ●

SQMR

QMR method [Freund, 1992]:

Let the the system $Ax = b$, with A nonsymmetric.

Nonsymmetric Lanczos:
$AV_m = V_{m+1}T_m$ with V_m **non-orthonormal basis** of $\mathcal{K}_m(A, r_0)$

At each iteration, we search y such as $x_m = x_0 + V_m y$

$$b - Ax_m = V_{m+1}(\|r_0\|e_1 - T_m y)$$

$$\text{QMR} \;\Rightarrow y = \text{argmin}\|\|r_0\|e_1 - T_m y\|$$

SQMR

SQMR method [Freund, 1995]:
If we know J nonsingular such as $A^T J = JA$, the Lanczos algorithm is simplified.

Application: Right preconditioning with H

$$Ax = b \Leftrightarrow AH\tilde{x} = b, \;\; x = H\tilde{x}$$

If A is symmetric, $H(AH) = (AH)^T H$ and we take $J = H$

Interest: **symmetric preconditioner** , contrary to MINRES or SYMMLQ method (requiring a SPD preconditioner)

Application of the preconditioner, computational cost

Construction of H

- Computational cost for the projection of S thanks to the sign matrix
- Construction of Z_+ (resp. Z_-) K-conjugate basis (resp. $-K$) of $range(S_+)$ (resp. $range(S_-)$)
  computational cost: $2n(k_+^2 + k_-^2)$ where $k = k_+ + k_-$, and k matrix-vector products by K

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Application of H

Let $Z = [Z_+, Z_-]$, $\overline{Z} = [Z_+, -Z_-]$ and $Y = [KZ_+, -KZ_-]$:

$$Hq = (I_n - ZY^T)H_0(I_n - YZ^T)q + Z\overline{Z}q$$

Computational cost: $10kn$ and one matrix-vector product by H_0
Storage: $2k$ vectors of length n

Main steps

Framework: M SPD, $K = M^{\frac{1}{2}} A M^{\frac{1}{2}}$ and $H_0 = I$.

Algorithm:

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- SQMR or GMRES for K right preconditioned by H

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Remark: It is possible to reuse directly H in a sequence framework (reduction of the computational cost)

Sommaire

Academic test problem

- Stationary Navier-Stokes problem: "Lid driven cavity" in a square domain (from IFISS library)
- $Q_2 - Q_1$ finite element
- Picard method to linearize the problem

Sequence: $$\begin{pmatrix} K + N_i & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} u_i \\ p_i \end{pmatrix} = \begin{pmatrix} f_i \\ g_i \end{pmatrix}$$

Stokes with multiple right-hand sides, $n = 2467$

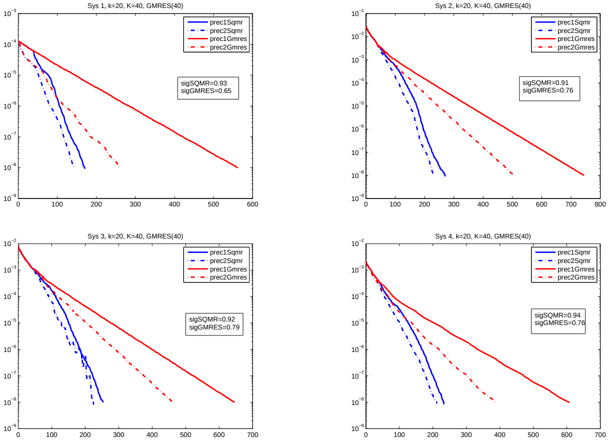
Sequence: $Ax_i = b_i \Leftrightarrow \begin{pmatrix} K & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} u_i \\ p_i \end{pmatrix} = \begin{pmatrix} f_i - N_i u_{i-1} \\ g_i \end{pmatrix}$

Characteristics:

- $n = 2467$, 4 systems to solve.
- $M = LL^T$ Incomplete Cholesky factorization of $\begin{pmatrix} K & 0 \\ 0 & Q \end{pmatrix}$ with Q the pressure mass matrix (drop tolerance 10^{-2}).
- H is "built" during the solution of the 1st system. It is also applied to the 3 other systems.
- S contains k Ritz vectors (projected) corresponding to the k Ritz values with smallest modulus.
- The sign matrix of $L^{-1}AL^{-T}$ is exact (with signm.m from the matrix function toolbox by N.Higham)

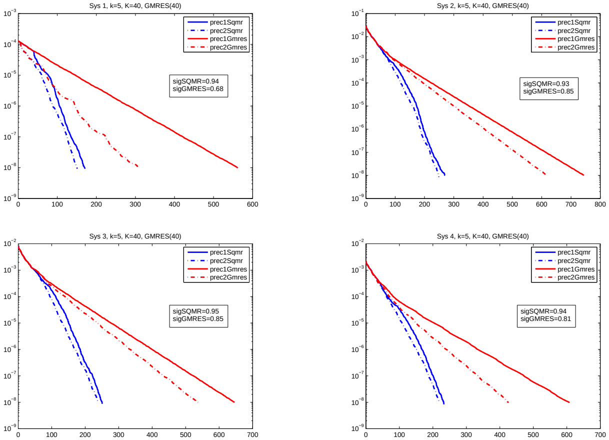
Stokes with multiple right-hand sides, $n = 2467$

Convergence curves of relative residual $\frac{\|b - Ax_i\|_2}{\|b\|_2}$ with $k = 20$ ($K = 40$):



Stokes with multiple right-hand sides, $n = 2467$

Convergence curves of relative residual $\frac{\|b - Ax_i\|_2}{\|b\|_2}$ with $k = 5$ ($K = 40$):



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- Generalization to the nonsymmetric case
- Non-standard inner product

Thank you for your attention!

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