Theoretical and numerical study of a preconditioner for saddle-point systems

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Outline

1 Motivations

2 Preconditioning of saddle-point system

3 Spectral analysis

4 Numerical results

5 Conclusions
The Stokes problem

\[ \begin{cases} -\nu \Delta u + \text{grad } p = f \quad \text{in } \Omega \\ \text{div } u = 0 \end{cases} \]

- $u$ velocity, $p$ pressure
- used in computational fluid dynamics, in climate forecasting, ...

When discretizing the equations:

\[
\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}
\]

$A \in \mathbb{R}^{n \times n}$ symmetric positive definite, $B \in \mathbb{R}^{m \times n}$.

With a time-dependent problem, we have a sequence of saddle-point...
How to solve this system?

- **Direct method**: compute the exact solution by factorizing the matrix
  - need access to matrix coefficients
  - can be expensive when large-scale systems

- **Iterative methods**: build a sequence of vectors \((x_k)\) converging to the exact solution of the systems
  - does not need the access to matrix coefficients
  - preconditioning to accelerate the convergence of the method

In our case, we consider a large-scale system solved by a preconditioned Krylov subspace method.
Preconditioning

To precondition a system we solve a system with the same solution, supposing this system is easier and quicker to solve than the initial one.

An efficient preconditioning matrix must be:

- low cost to compute and low cost in memory
- low cost to use
- accelerate the convergence of the method

The preconditioned system must have less distinct eigenvalues than the initial one.

It is difficult to precondition a saddle-point system because it is an indefinite system with eigenvalues in the complex plane.
Well-known preconditioners

\[ A = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \]

- Block preconditioners ([Murphy, Golub, Wathen, 2000], [Klawonn, 1998], [Ruiz, Sartenaer, Tannier, 2014])

\[ P_d = \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix}, \quad P_t = \begin{pmatrix} A & B^T \\ 0 & S \end{pmatrix} \]

\( S = -B^T A^{-1} B \) the Schur complement.

- Krylov method converge in at most 3 or 2 iterations
- \( S \) can be expensive to compute
- Constraint preconditioner ([Keller, Gould, Wathen, 2000])

\[ P_c = \begin{pmatrix} G & B^T \\ B & 0 \end{pmatrix} \]

\( S = -BG^{-1}B^T \) the Schur complement of \( P_c \)

- 1 is eigenvalue with multiplicity 2m
We solve a saddle-point system with a preconditioned Krylov method (**first-level preconditioner**).

We use an approximation of a well-known preconditioner.

The preconditioned system has a **cluster of eigenvalues around isolated eigenvalues**, other eigenvalues out of this cluster.

We want to find another preconditioner (**second-level preconditioner**) to reduce the number of eigenvalues out of the cluster.
Case of a symmetric positive definite system [Gratton, Sartenaer, Tshimanga, 2011], [Morales, Nocedal, 2000]

- We want to solve the system $Ax = b$ with $A$ symmetric positive definite
- We have secant equations $y_k = A s_k$
- We have a preconditioner $M$ (first-level preconditioner)
- We use the BFGS update formula to compute the preconditioner (second-level preconditioner):

$$H_k = [I_n - S(S^T AS)^{-1} S^T A]M[I_n - AS(S^T AS)^{-1} S^T] + S(S^T AS)^{-1} S^T.$$

- The update can be seen as variational approach
- At least $k$ eigenvalues equal to 1
- The other part of the spectrum of $AH_k$ is not expanded
Objectives

- We want to solve the saddle-point system:

\[
\begin{pmatrix}
A & B^T \\
B & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= 
\begin{pmatrix}
b \\
c
\end{pmatrix}
\]

with a Krylov method.

- We can choose one of the well-known preconditioners according to the structure of the problem (first-level preconditioner)

- We want to modify the existing preconditioner from secant equations (second-level preconditioner)

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- We want to modify the existing preconditioner from secant equations (*second-level preconditioner*).

Objectives:

Find a second-level preconditioner using a variational approach with a Frobenius norm.
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Second-level preconditioning

We have the matrix-vector product:

\[
\begin{pmatrix}
A & B^T \\
B & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{x} \\
\tilde{y}
\end{pmatrix}
=
\begin{pmatrix}
b \\
c
\end{pmatrix}.
\]
Second-level preconditioning

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\]

\[
[\tilde{x}^T \quad \tilde{y}^T]^T:
\]

- can be an eigenvector of \( A \),
- can satisfy the Arnoldi relation,
- can be an Harmonic-Ritz vector of \( A \).
Second-level preconditioning

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\[
[\tilde{x}^T \quad \tilde{y}^T]^T:
\]
- can be an eigenvector of $A$,
- can satisfy the Arnoldi relation,
- can be an Harmonic-Ritz vector of $A$.

We have the first-level preconditioner $P_0 = \begin{pmatrix}
\tilde{A} & \tilde{B}^T \\
\tilde{B} & 0
\end{pmatrix}$. We want to find the second-level preconditioner $P_1$ such that

\[
\underbrace{\begin{pmatrix}
\tilde{A} & \tilde{B}^T \\
\tilde{B} & 0
\end{pmatrix} + \begin{pmatrix}
\Delta A & \Delta B^T \\
\Delta B & 0
\end{pmatrix}}_{P_1}
\begin{pmatrix}
\tilde{x} \\
\tilde{y}
\end{pmatrix}
= \begin{pmatrix}
b \\
c
\end{pmatrix}.
\]

How to choose $\Delta A, \Delta B$?
Second-level preconditioning

\[
\begin{pmatrix}
\tilde{A} + \Delta A & \tilde{B}^T + \Delta B^T \\
\tilde{B} + \Delta B & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{x} \\
\tilde{y}
\end{pmatrix}
= 
\begin{pmatrix}
\tilde{b} \\
\tilde{c}
\end{pmatrix}
\]

\[
\Rightarrow 
\begin{pmatrix}
\Delta A\tilde{x} + \Delta B^T\tilde{y} = b - \tilde{A}\tilde{x} - \tilde{B}^T\tilde{y} \\
\Delta B\tilde{x} = c - \tilde{B}\tilde{x}
\end{pmatrix}
\]

We define \( r_b, r_c, w \) as:

\[
\begin{align*}
    r_b &= b - \tilde{A}\tilde{x} - \tilde{B}^T\tilde{y}, \\
    r_c &= c - \tilde{B}\tilde{x}, \\
    w &= r_b - \Delta B^T\tilde{y}
\end{align*}
\]

We have:

\[
\begin{align*}
    \Delta A\tilde{x} + \Delta B^T\tilde{y} &= r_b \\
    \Delta B\tilde{x} &= r_c
\end{align*}
\]
Second-level preconditioning

\[
\begin{pmatrix}
\tilde{A} + \Delta A & \tilde{B}^T + \Delta B^T \\
\tilde{B} + \Delta B & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{x} \\
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We have:

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\begin{align*}
\Delta A\tilde{x} + \Delta B^T\tilde{y} &= r_b \\
\Delta B\tilde{x} &= r_c
\end{align*}
\]

→ We want to find the least Frobenius norm update!
Any solution of the equations, $\Delta A$ and $\Delta B$ can be written as:

\[
\Delta B = r_c \tilde{x}^\dagger + ZP_\perp \tilde{x}, \\
\Delta A = w \tilde{x}^\dagger + (w \tilde{x}^\dagger)^T - (\tilde{x}^\dagger)^T w^T P_\tilde{x} + P_\perp \tilde{x}^\dagger TP_\perp \tilde{x}
\]

with $T \in \mathbb{R}^{n \times n}, Z \in \mathbb{R}^{m \times m}$.

We want to find $Z, T$ that minimize

\[
\min_{Z, T} \|\Delta A\|_F^2 + \|\Delta B\|_F^2
\]

The least-norm solution is:

\[
T = 0, \ Z = 2P_\perp (r_b \tilde{x}^\dagger)(\tilde{y} \tilde{x}^\dagger)^T (I_m + 2(\tilde{y} \tilde{x}^\dagger)(\tilde{y} \tilde{x}^\dagger)^T)^{-1}
\]

The second-level preconditioner can be written as

\[
P_1 = P_0 + FG^T, \ F, G^T \in \mathbb{R}^{(n+m) \times 5}
\]

Using the Sherman-Morrison formula we have:

\[
P_1^{-1} = P_0^{-1} - P_0^{-1}F \left(I_5 + G^T P_0^{-1} F \right)^{-1} G^T P_0^{-1}.
\]
Outline

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2. Preconditioning of saddle-point system
3. Spectral analysis
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Spectral analysis

A clustered spectrum often results in rapid convergence when using GMRES.

We define:

- $\mathcal{A} = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$, $A$ symmetric positive definite, rank $B = m$,
- $\lambda^A_{\text{min}}, \lambda^A_{\text{max}}$ the minimal and maximal eigenvalues of $A$,
- $\sigma_{\text{min}}, \sigma_{\text{max}}$ the smallest non-zero and the maximum singular values of $B$.

We know that [Rusten, Winther, 1992] the eigenvalues of $\mathcal{A}$ are contained in $\mathcal{I}_0^- \cup \mathcal{I}_0^+ \subset \mathbb{R}$:

$$
\mathcal{I}_0^- = \left[ \frac{1}{2} \left( \lambda^A_{\text{min}} - \sqrt{(\lambda^A_{\text{min}})^2 + 4\sigma^2_{\text{max}}} \right), \frac{1}{2} \left( \lambda^A_{\text{max}} - \sqrt{(\lambda^A_{\text{max}})^2 + 4\sigma^2_{\text{min}}} \right) \right],
$$

$$
\mathcal{I}_0^+ = \left[ \lambda^A_{\text{min}}, \frac{1}{2} \left( \lambda^A_{\text{max}} + \sqrt{(\lambda^A_{\text{max}})^2 + 4\sigma^2_{\text{max}}} \right) \right].
$$
Spectral analysis

We suppose:

- $\mathcal{P}_0 = I_{n+m}$

- $\mathcal{P}_1$ is block diagonal symmetric,

\[
\begin{bmatrix}
\tilde{x}^T & \tilde{y}^T
\end{bmatrix}^T = 
\begin{bmatrix}
u^T & v^T
\end{bmatrix}^T
\] is an eigenvector of $A$ associated to the eigenvalue $\lambda$.

We define $P_u, P_v$ the orthogonal projectors onto the range of $u$ and $v$. We can deduce that:

\[
\mathcal{P}_1 = \begin{pmatrix}
I_n + (\lambda - 1)P_u & 0 \\
0 & I_m + (\lambda - 1)P_v
\end{pmatrix}.
\]
Spectral analysis

Let’s denote $\mu$ an eigenvalue of $P_1^{-1}A$ and $(x^Ty^T)^T$ its associated eigenvector.

**Theorem 1**

Let’s denote $x = x_u + x_{u\perp}$, $y = y_v + y_{v\perp}$, $x_u \in \text{Span}(u)$, $x_{u\perp} \in \text{Span}(u)\perp$, $y_v \in \text{Span}(v)$, $y_{v\perp} \in \text{Span}(v)\perp$. The following equation is satisfied:

$$
\mu^2 \left( \|x_{u\perp}\|^2_2 + \lambda \|x_u\|^2_2 \right) - \mu x^T Ax - \|y_{v\perp}\|^2_2 - \frac{1}{\lambda} \|y_v\|^2_2 = 0
$$

**Particular case (assuming $x_{u\perp} = 0$, $y_{v\perp} = 0$)**

1 and $-\frac{\|v\|^2_2}{\|u\|^2_2}$ are eigenvalues of $P_1^{-1}A$. 

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Spectral analysis

We suppose $\lambda > 0$. Let’s denote

- $B \begin{pmatrix} Y & Z \end{pmatrix} = \begin{pmatrix} L & 0 \end{pmatrix}$ an LQ factorization of $B$, with $Z$ a basis of the nullspace of $B$.
- $\lambda^Y_{\text{min}}, \lambda^Y_{\text{max}}$ the minimal and maximal eigenvalues of $Y^T A Y$.
- $\sigma^Y_{\text{min}}, \sigma^Y_{\text{max}}$ the smallest non-zero and the maximum singular values of $Y$.

**Theorem 2**

The eigenvalues of $P^{-1}_1 A$ are contained in $I^-_1 \cup I^+_1 \subset \mathbb{R}$:

$I^-_1 = \left[ \frac{\lambda^A_{\text{min}} - \sqrt{\left(\lambda^A_{\text{min}}\right)^2 + 4\left(\sigma^B_{\text{max}}\right)^2}}{2\min(1, \lambda)}, \frac{\lambda^Y_{\text{max}} - \sqrt{\left(\lambda^Y_{\text{max}}\right)^2 + 4\left(\sigma^B_{\text{min}}\right)^2\left(\sigma^Y_{\text{max}}\right)^2}}{2\max(1, \lambda)\left(\sigma^Y_{\text{max}}\right)^2} \right]$, 

$I^+_1 = \left[ \frac{\lambda^A_{\text{min}}}{\max(1, \lambda)}, \frac{\lambda^A_{\text{max}} + \sqrt{\left(\lambda^A_{\text{max}}\right)^2 + 4\left(\sigma^B_{\text{max}}\right)^2}}{2\min(1, \lambda)} \right]$. 

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Numerical illustration

\[ A = \begin{pmatrix} 17 & -2 & 0 \\ -2 & 16 & 0 \\ 0 & 0 & 0.8 \end{pmatrix}, \quad B^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \]

\[ l_0^- = [-0.6770; -0.0537], \quad l_0^+ = [0.8; 18.6153] \]

\[ \text{Sp}(\mathcal{A}) = \{-0.0689; -0.0537; 0.8; 14.5074; 18.6153\}. \]

<table>
<thead>
<tr>
<th>\lambda</th>
<th>\mathcal{I}_1^-</th>
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<th>\text{Sp}(\mathcal{A} \mathcal{P}_1^{-1})</th>
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The Stokes problem

- Stokes problem from Ifiss library
- channel domain
- natural boundary outflow
- stabilization parameter : $1/4$
- uniform streamline

We solve the first system with the preconditioner $\mathcal{P}_0 : \mathcal{A}_0 = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$.

We solve a second system with the preconditioner $\mathcal{P}_1 : \mathcal{A}_1 = \begin{pmatrix} A + \tilde{A} & B^T \\ B & 0 \end{pmatrix}$.

Both systems are solved using a GMRES method with a $10^{-8}$ tolerance.
- grid: 32*32 points
- $P_0 = I_{n+m}$
- $P_1$ computed from eigenvectors of $A$

$\rightarrow$ using eigenvectors associated to the minimum modulus eigenvalues may accelerate the GMRES convergence but
- eigenvectors are expensive to compute
- expensive in memory
- cannot be applied with larger grid
grid: 32*32 points

$P_0 = I_{n+m}$

$P_1$ computed from Harmonic-Ritz vectors of $A$

→ Using 30 or more Harmonic-Ritz vectors of $A$ accelerates the GMRES convergence but the grid is too small to gain flops.
grid : 256*256 points

\[ P_0 = \begin{pmatrix} I_n & B^T \\ B & 0 \end{pmatrix} \]

→ gain 50% of flops when using 10 Harmonic-Ritz vectors associated to values of minimum modulus, low cost in memory
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Conclusions

- We have derived a second-level preconditioner using a low-rank least Frobenius norm update for a saddle-point system

\[ A = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \]

- We have derived a second-level preconditioner using a low-rank least Frobenius norm update for a generalized saddle-point system

\[ A = \begin{pmatrix} A & E^T \\ F & C \end{pmatrix} \]

- We have provided a spectral analysis for a specific case where \( A \) is symmetric positive definite, \( B^T \) is full-rank.

- We have analyzed the performance of the second-level preconditioner by using a Stokes problem.
Develop a methodology to analyze the spectrum of $P_1^{-1}A$ with other first-level preconditioner and other vectors to compute $P_1$.

Study the sharpness of the bounds given by the spectral analysis.

Solve a sequence of system by using a time-dependent Stokes problem.
Thank you for your attention!
Harmonic-Riz values of $A$, grid of size 32*32
Harmonic-Ritz values of $\mathcal{A}$, grid of size 256*256
The Arnoldi relation

We want to solve the system $Ax = b$, $A \in \mathbb{R}^{n \times n}$.

Let’s denote $V_m \in \mathbb{R}^{n \times m}$ the matrix containing the vectors $v_1, \ldots, v_m$ computed by the Arnoldi procedure. $H_m$ the Hessenberg matrix of size $(m + 1) \times m$.

The Arnoldi relation

$$AV_m = V_{m+1}H_m.$$ 

If we use the right preconditioner $P$, let’s denote $Z_m = P^{-1}V_m$ and we have :

The Arnoldi relation

$$AZ_m = V_{m+1}H_m.$$
Harmonic-Ritz vector

Definition

A pair \((\lambda, x)\) is called an Harmonic-Ritz pair of \(A\) with respect to the subspace \(\mathcal{L}\) if \(x \in \mathcal{L}\) and \((Ax - \lambda x) \perp A\mathcal{L}\).

In our case \(A = AZ_m V_m^\dagger\) and \(\mathcal{L} = \text{Span}(V_m)\).