Numerical Analysis in Optimal Control Problem for Aircraft Trajectories

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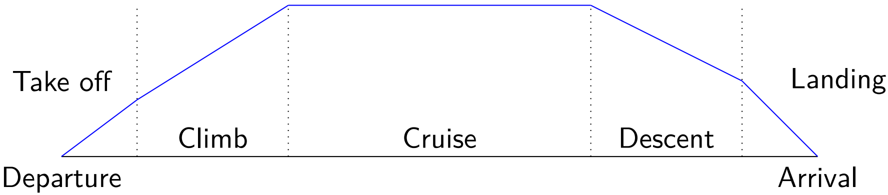
Summary

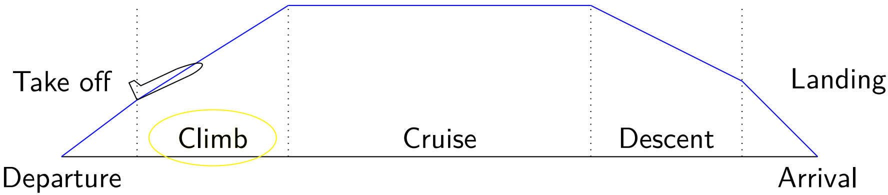
Aircraft trajectories include several flight phases with a lot of constraints



Each phase gets some constraints from :

- Air traffic control
- Flight enveloppe
- Historical practices

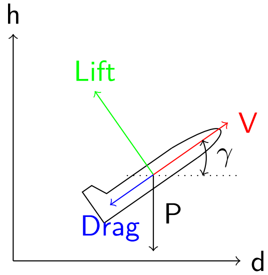
Climb phase for a middle-haul aircraft will be considered in this study



Constraints to be considered for climbing phase :

- maximum speed, or limited climb rate
- climb procedure to follow

Aircraft dynamics :



$$\frac{\mathrm{d}h}{\mathrm{d}t} = V\sin(\gamma)$$
$$\frac{\mathrm{d}d}{\mathrm{d}t} = V\cos(\gamma)$$
$$m\frac{\mathrm{d}V}{\mathrm{d}t} = \varepsilon T_{max}(h) - \overbrace{\frac{1}{2}\rho(h)SV^2C_D(C_L)}^{\text{Drag}} - mg\sin(\gamma)$$
$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\varepsilon C_s(V)T_{max}(h)$$
$$mV\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \underbrace{\frac{1}{2}\rho(h)SV^2C_L}_{\text{Lift}} - mg\cos(\gamma)$$

h : altitude	γ : air slope	C_L, C_D : lift,drag coefficients
d : longitudinal distance	C_s : fuel flow	ε : thrust ratio
V : air speed	T_{max} : maximal Thrust	S : wing area
m : weight	ρ : air density	g : gravitational constant

Atmospheric model :

ISA (*International Standard Atmosphere*) model

$$\begin{cases} T = T_0 - \beta h \\ P = P_0 \left(\frac{T}{T_0}\right)^{\frac{g}{\beta R}} \end{cases} \rightarrow \rho = \frac{P}{RT},$$ R is the specific constant of air.

Aircraft data model :

BADA model

$T_{\max} = C_{T_1}\left(1 - \frac{h}{C_{T_2}} + C_{T_3} h^2\right)$ is the maximal thrust

$C_s = C_{s_1}\left(1 + \frac{V}{C_{s_2}}\right)$ is the fuel flow

$C_D = C_{D1} + C_{D2}.C_L^2$ is the drag coefficient

- state : $x = (h, d, v, m, \gamma)$
- control : $u = (\varepsilon, C_L)$
- parameters : $\omega = (S, g, C_{T_1}, C_{T_2}, C_{T_3}, C_{D_1}, C_{D_2}, C_{s_1}, C_{s_2}, R, T_0, \beta, P_0)$
- auxiliary functions : $\theta(x) = (\theta_1(x), \theta_2(x), \theta_3(x), \theta_4(x))$

h is the altitude

d is the longitudinal distance

V is the air speed

γ is the air slope

m is the weight

ε is the thrust ratio

C_L is the lift coefficient

Then we write the dynamics :

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,u) = f_0(x) + u_1 f_1(x) + u_2 f_2(x) + u_2^2 f_3(x)$$

With

$$\left|\begin{array}{l} f_0 = \left(x_3 \sin(x_5), x_3 \cos(x_5), -\omega_6 \theta_3(x) - \omega_2 \sin(x_5), 0, -\frac{\omega_2}{x_3} \cos(x_5)\right)^T \\ f_1 = \left(0, 0, \frac{\theta_1(x)}{x_4}, -\theta_1(x)\theta_2(x), 0\right)^T \\ f_2 = \left(0, 0, 0, 0, \frac{\theta_3(x)}{x_3}\right)^T \\ f_3 = (0, 0, -\omega_7 \theta_3(x), 0, 0)^T \end{array}\right.$$

On this study, we consider 4 constraints concerning :

- the air slope, $x_{5_{\min}} \leqslant x_5 \leqslant x_{5_{\max}}$
- the CAS, $\phi(x) \leqslant \phi_{\max}$
- the Mach, $\psi(x) \leqslant \psi_{\max}$

where

$$\begin{cases} \phi(x) = \sqrt{\dfrac{2\omega_{11}\omega_{10}}{\omega_{14}}\left(\left(\dfrac{\theta_5(x)}{\omega_{13}}\left(\left(\dfrac{\omega_{14}x_3^2}{2\omega_{10}\theta_4(x)}+1\right)^{\frac{1}{\omega_{14}}}-1\right)+1\right)^{\omega_{14}}-1\right)} \\ \psi(x) = \dfrac{x_3}{\sqrt{\omega_{15}\omega_{10}\theta_4(x)}} \end{cases}$$

These considerations lead us to solve an optimal control problem (OCP) written in a *Mayer's formulation* and defined by :

$$
\left|
\begin{array}{l}
\min\limits_{(t_f,x,u)} t_f \\
\dfrac{\mathrm{d}x}{\mathrm{d}t}(t) = f(x(t),u(t)),\ \ t \in [0,t_f] \text{ a.e.} \\
u(t) \in \mathcal{U} = \left\{u = (u_1,u_2) \in \mathbb{R}^2, u_i \in [u_{i,\min}, u_{i,\max}], i = 1,2\right\} \\
x(t) \in \mathcal{X} = \mathbb{R}^5 \\
c(x(t)) \leqslant 0, \quad t \in [0,\ t_f] \\
x(0) = x_0,\ \ x_f \in \mathcal{X}_f = \{x(t_f) \in \mathcal{X}, b_f(x(t_f)) = 0\} \subset \mathcal{X}
\end{array}
\right.
$$

$$
b_f(x) = \begin{pmatrix} x_{1_f} - x_1(t_f) \\ x_{2_f} - x_2(t_f) \\ x_{3_f} - x_3(t_f) \\ x_{5_f} - x_5(t_f) \end{pmatrix},\ c(x) = \begin{pmatrix} x_{5_{\min}} - x_5 \\ x_5 - x_{5_{\max}} \\ \phi(x) - \phi_{\max} \\ \psi(x) - \psi_{\max} \end{pmatrix}
$$

$$(OCP)\ :\ \begin{cases} \min\ t_f \\ \frac{\mathrm{d}x}{\mathrm{d}t}(t)=f(x(t),u(t)),\ x(0)-x_0=0,\ b_f(x(t_f))=0,\ u\in\mathcal{U},\ t\in[0,t_f]\ \text{a.e} \end{cases}$$

Hamiltonian : H(p(t),x(t),u(t))= $\langle p(t),f(x(t),u(t))\rangle+\langle\eta(t),c(x(t))\rangle$

• Necessary conditions (PMP) : If (t_f^*, x^*, u^*) is optimal then $\exists p^* \in \mathrm{AC}\left([0,t_f],(\mathbb{R}^n)^*\right)$, $(p^*, p^0) \neq (0,0)$ such as a.e :

$$\left|\begin{aligned} \dot{x}(t) &= \frac{\partial H}{\partial p}(p^*(t), x^*(t), u^*(t)) \\ \dot{p}(t) &= -\frac{\partial H}{\partial x}(p^*(t), x^*(t), u^*(t)) - \eta(t)\frac{\partial c}{\partial x}(x(t)) \\ H(p^*(t), x^*(t), u^*(t)) &= \max_{u\in\mathcal{U}} H(p^*(t), x^*(t), u) \end{aligned}\right.$$

$$(OCP) \;:\; \begin{cases} \min \; t_f \\ \frac{\mathrm{d}x}{\mathrm{d}t}(t) = f(x(t),u(t)),\; x(0)-x_0=0,\; b_f(x(t_f))=0,\; u\in\mathcal{U},\; t\in[0,t_f] \text{ a.e} \end{cases}$$

Hamiltonian : H(p(t),x(t),u(t))= $\langle p(t), f(x(t),u(t))\rangle + \langle \eta(t), c(x(t))\rangle$

With the transversality conditions :

$$\begin{aligned} p_4^*(t_f) &= 0 \\ H(p^*(t_f), x^*(t_f), u^*(t_f)) &= -\,p^0 \end{aligned}$$

At any time t in $[0,\; t_f]$, we have $\eta_i(t)c_i(x(t)) = 0$ and $\eta_i(t) \leqslant 0$

State constraints bring some conditions on the junction or contact time τ :

$$\begin{aligned} H[\tau^+] &= H[\tau^-] \\ p(\tau^+) &= p(\tau^-) - \nu_\tau c'(x(\tau)),\; \nu_\tau \leqslant 0 \end{aligned}$$

Definition We called an extremal the quadruplet $(x(\cdot), p(\cdot), u(\cdot), \eta(\cdot))$ which satisfy the necessary conditions.

We consider an extremals with no active state constraints, then

- $\eta(\cdot) \equiv 0$,
- control $u = (u_1, u_2)$ is defined by
  - $u_1(t) = u_{1,\max}$ if $H_1 > 0$, else $u_{1,\min}$,
  - $u_2(t) = -\frac{H_2}{2H_3}$ if $H_3 < 0$, else $\begin{cases} u_{2,\max}, & \text{if } H_2+2u_2H_3>0 \\ u_{2,\min}, & \text{else} \end{cases}, \forall t \in [0, t_f]$,
- $x(\cdot)$ and $p(\cdot)$ are deduced from necessary condtions.

Remark $H_i = \langle p, f_i \rangle$ is called the Hamiltonian lift of f_i

Definition We define the order m of the constraint c as the first integer such that $\frac{\partial c}{\partial u} = \frac{\partial \dot{c}}{\partial u} = \ldots = \frac{\partial c^{(m-1)}}{\partial u} = 0$ and $\frac{\partial c^{(m)}}{\partial u} \neq 0_{\mathbb{R}^2}$.

All constraints of this probem are states constraints of order 1.
We consider here that only the constraint c_1 is active, then
$c_1(x) = 0 = x_{5,\min} - x_5$.
As c_1 is of order 1, $c_1 = \dot{c}_1 = 0$, $\frac{\partial c_1}{\partial u} = 0$ and $\frac{\partial \dot{c}_1}{\partial u} = (0, \frac{\theta_3}{x_3}) \neq 0$.

These information are usefull to determine the value of the quadruplet $(x(\cdot), p(\cdot), u(\cdot), \eta(\cdot))$ on this constraint extremal.

– As $\dot{c}_1$ depends on u_2, the constrained control $u = (u_1, u_2)$ is defined by
  - $u_1(t) = u_{1,\max}$ if $H_1 > 0$, else $u_{1,\min}$,
  - $u_2(t) = \frac{\omega_2}{\theta_3} \cos(x_{5,\min}), \forall t \in [0, t_f]$,

– We assume that u_2 belongs to $(u_{2,\min}, u_{2,\max})$, then
$$\dot{\varphi}_2 = 0 = \{H, \varphi_2\} = \{H, H_2\} + 2u_2\{H, H_3\} - \eta \underbrace{c_1'(f_2 + 2u_2 f_3)}_{=\frac{\partial \dot{c}_1}{\partial u_2} \neq 0} + 2H_3 u_2' f$$

$$\text{and } \eta = \frac{\{H, H_2\} + 2u_2\{H, H_3\} + 2H_3 u_2' f}{c_1'(f_2 + 2u_2 f_3)}$$

– $x(\cdot)$ and $p(\cdot)$ are then deduced from necessary condtions.

Remark

- If control u_i belongs to $(u_{i,\min}, u_{i,\max})$ then $\varphi_i := \frac{\partial H}{\partial u_i} \equiv 0$ (maximisation condition)
- $\{H, H_i\} = \langle p, [f, f_i] \rangle$ is called the Poisson bracket, where $[f, f_i]$ is the Lie bracket between f and f_i.

Bocop is a software using Direct Methods to solve Optimal Control Problem. This software transform an OCP into a Non linear Problem which is solved by the well known solver IPOPT.

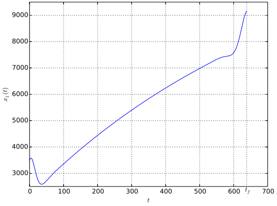
Data :

$$\left| \begin{array}{ll} u_{1min} = 0.3, & u_{1max} = 1 \\ u_{2min} = 0, & u_{2max} = 1.6 \end{array} \right., \quad \left| \begin{array}{l} x_0 = (3480, 0, 145, 64000, 0.25)^T \\ x_f = (9144, 150\ 000, 236.5, \text{free}, 0)^T \end{array} \right.$$

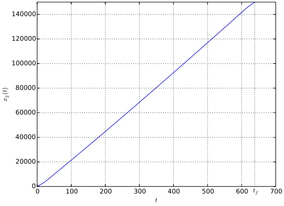
Numerical Test :

- Obtained with tolerance of 10^{-10}
- Use of Euler Implicit scheme (first order), with 500 discretised points

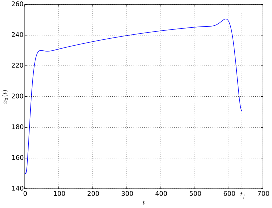
Unconstrained Case



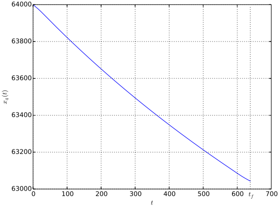
x_1



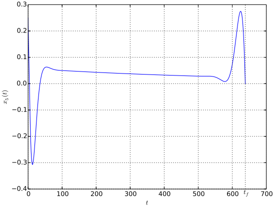
x_2



x_3

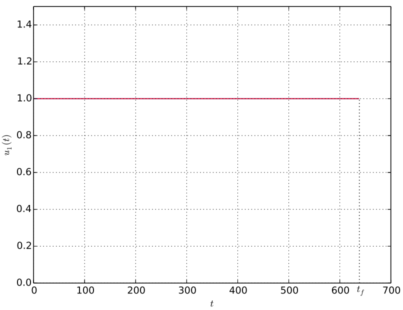


x_4

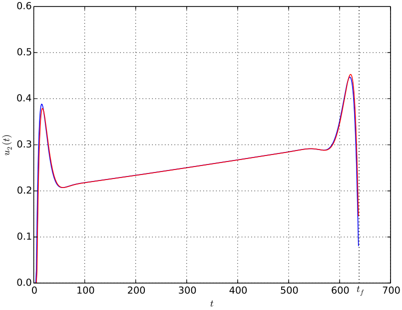


x_5

Unconstrained Case



u_1

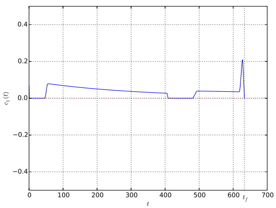


u_2

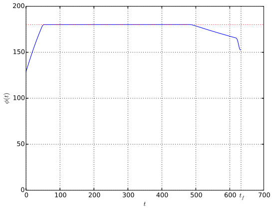
legend :

- postcomputed data
- data from Bocop

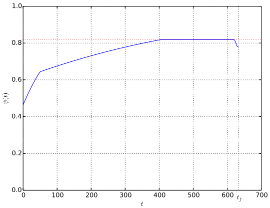
Constrained Case



c_1

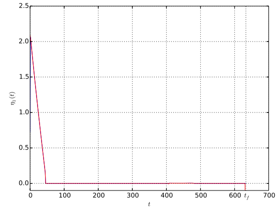


c_3

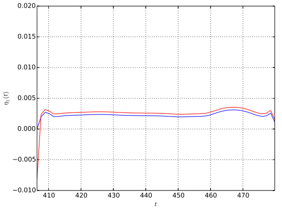


c_4

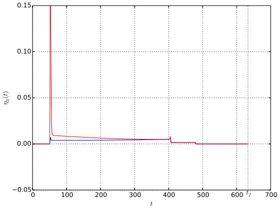
Constrained Case



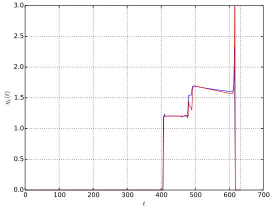
η_1



zoom of η_1

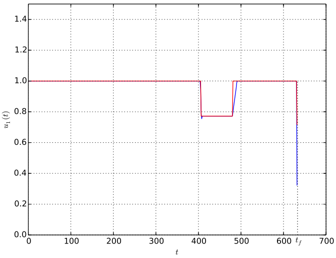


η_3

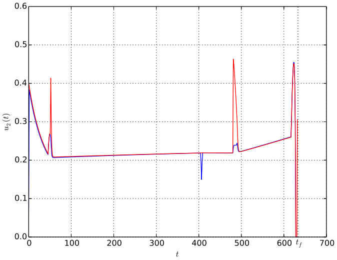


η_4

Constrained Case



u_1



u_2

Conclusion The analysis of the results from direct methods by the indirect methods allows a better understanding of the behavior of the optimal solution.

Future Work

- Use indirect numerical methods as multiple shooting to solve the problem
- Initialise indirect methods using results from the direct methods
- Comparison with pratical procedure
- Analysis of others aircraft and other flight phases