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MfMax(v0 & v1): Method explanation manual[1]

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Contents

Chapter 1

Problem statement

1.1 The minimum fuel orbital transfer

We consider here the 3D orbital transfer of a satellite around the Earth in which we seek to minimize fuel consumption (maximization of final mass). The initial orbit is very eccentric and the final one is geostationnary:

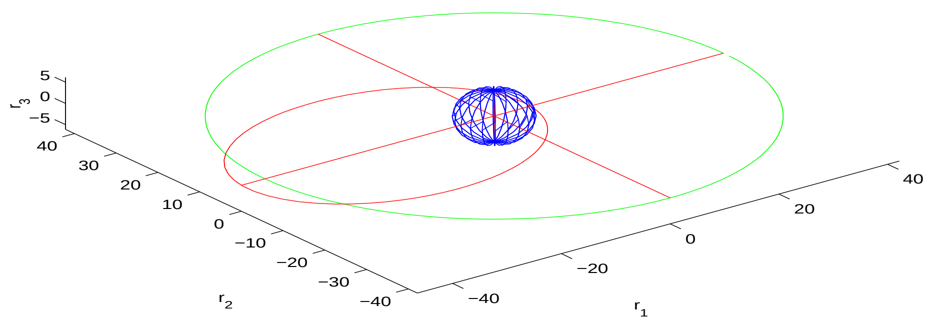


Figure 1: LOW to GTO transfer

We express the position of the satellite in the Gauss coordinates:

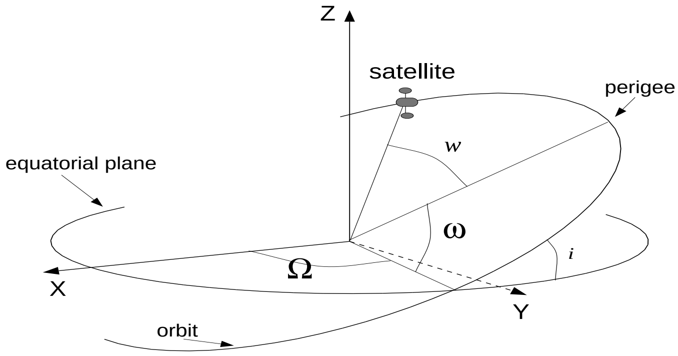


Figure 2: Orbital coordinates

with

- P and e: ellipse parameter and eccentricity
- w: true anomaly
- Ω: ascending node longitude
- ω: argument of perigee
- i: inclination towards equatorial plane

Let us now define our state variables in $\mathbf{R^7}$:

- Variables described oscullating ellipse of the trajectory:
  - Orbit parameter P
  - Eccentricity vector $[e_x, e_y]$, in the orbit plane, oriented towards perigee
  - Rotation vector $[h_x, h_y]$, in the equatorial plane, colinear to the intersection of orbit and equatorial planes
  - True longitude L
- Mass variable m

The state is then:

$$x = [P, e_x, e_y, h_x, h_y, L, m] \in \mathbf{R}^7$$

The optimal control problem we solve is the following (see [1] or [2]):

$$
\left\{
\begin{array}{lll}
max\ m(t_f) & & \\
\dot{P}(t) & = & \frac{2}{m(t)}\sqrt{\frac{P^3(t)}{\mu_0}}\frac{u_2(t)}{Z(t)} \\
\dot{e}_x(t) & = & \frac{1}{m(t)}\sqrt{\frac{P(t)}{\mu_0}}\frac{1}{Z(t)}[Z(t)sin(L(t))u_1(t) + A(t)u_2(t) - \\
 & & e_y(t)(h_x(t)sin(L(t)) - h_y(t)cos(L(t)))u_3(t)] \\
\dot{e}_y(t) & = & \frac{1}{m(t)}\sqrt{\frac{P(t)}{\mu_0}}\frac{1}{Z(t)}[-Z(t)cos(L(t))u_1(t) + B(t)u_2(t) + \\
 & & e_x(t)(h_x(t)sin(L(t)) - h)y(t)cos(L(t)))u_3(t)] \\
\dot{h}_x(t) & = & \frac{1}{2m(t)}\sqrt{\frac{P(t)}{\mu_0}}\frac{X(t)}{Z(t)}cos(L(t)).u_3(t) \\
\dot{h}_y(t) & = & \frac{1}{2m(t)}\sqrt{\frac{P(t)}{\mu_0}}\frac{X(t)}{Z(t)}sin(L(t)).u_3(t) \\
\dot{L}(t) & = & \sqrt{\frac{\mu_0}{P^3(t)}}Z^2(t) + \\
 & & \frac{1}{m(t)}\sqrt{\frac{P(t)}{\mu_0}}\frac{1}{Z(t)}(h_x(t)sin(L(t)) - h_y(t)cos(L(t)))u_3(t) \\
\dot{m}(t) & = & -\beta|u(t)| \\
\text{Where} & \left\{ \begin{array}{l} Z(t) = 1 + e_x(t)cos(L(t)) + e_y(t)sin(L(t)) \\ A(t) = e_x(t) + (1 + Z(t))cos(L(t)) \\ B(t) = e_y(t) + (1 + Z(t))sin(L(t)) \\ X(t) = 1 + h_x^2(t) + h_y^2(t) \end{array} \right. & \\
|u(t)| \leq T_{max},\ \forall t \in [0, t_f] & &
\end{array}
\right.
$$

$$
(version)\left\{ \begin{array}{l} \textbf{MfMax-v0: } \text{fixed } t_f \\ \textbf{MfMax-v1: } \text{free } t_f \end{array} \right.
$$

The state equation can be written as:

$$\dot{x}(t) = f(t, x, u)$$

Control is expressed in the ortho-radial frame attached to the satellite:

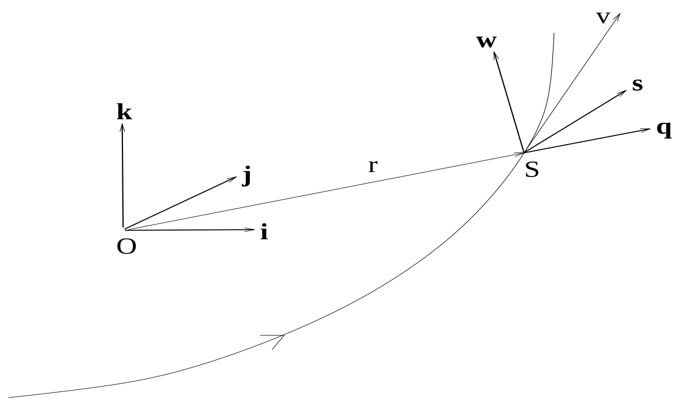


Figure 3: Control expression

$$u = (u_1, u_2, u_3) \in \mathbf{R}^3$$

Boundary conditions are given by:

$$x(0) = (P^0, e_x^0, e_y^0, h_x^0, h_y^0, L^0, m^0) \in \mathbf{R}^7$$

$$h(x(t_f)) = (P - P^f, e_x - e_x^f, e_y - e_y^f, h_x - h_x^f, h_y - h_y^f, L - L^f) \in \mathbf{R}^6$$

and

$$\begin{array}{lcllcl}
P^0 & = & 11625km & P^f & = & 42165km \\
e_x^0 & = & 0.75 & e_x^f & = & 0 \\
e_y^0 & = & 0 & e_y^f & = & 0 \\
h_x^0 & = & 0.0612 & h_x^f & = & 0 \\
h_y^0 & = & 0 & h_y^f & = & 0 \\
L^0 & = & \pi & L^f & = & \text{some multiplier of minimum longitude} \\
m^0 & = & 1500kg & m^f & free &
\end{array}$$

The two constants β and μ_0 are respectively taken equal to:

$$\begin{array}{rcl}
\beta & = & 1.42.10^{-5} km^{-1}.h \\
\mu_0 & = & 398600.47 km^3.s^{-2}
\end{array}$$

These physical datas, furnish by the French Space Agency correspond to a transfer from a high eccentricity trajectory to the geostationnary one.

We can rewrite the problem as follow:

$$(P_{m^f}) \left\{ \begin{array}{l}
Max\ m(t_f) \\
\dot{x}(t) = a(x(t)) + \frac{T_{Max}}{m(t)} B(x(t))u(t)\ ,\ \forall t \in [t_0, t^f] \\
\dot{m}(t) = -\beta T_{Max}\ |u(t)| \\
|u(t)| \leq 1,\ \forall t \in [t_0, t_f] \\
IC : x(t_0) = [11625; 0.75; 0; 0.0612; 0; \pi; 1500] \\
TC : x(t_f) = [42165; 0; 0; 0; 0; fixed; free] \\
t_0 = 0 \\
t_f = t_{min}^f.\ c_{tf} \text{ or free}
\end{array} \right.$$

Where t_{min}^f is the minimum transfer time corresponding to T_{max} and initial and terminal conditions (compute thanks **TfMin**, see [3]). c_{tf} is then a multipliar coefficient greater than 1 (otherwise transfer is not feasible) which caraterizes the problem.

Moreover, t_{min}^f is supposed (see [1],[2]) to be inversaly proportional to T_{max}. For our transfer's parameters, we take:

$$t_{min}^f(h.) T_{max}(N.) = 850$$

Besides, the actual criterion used for the maximization of the payload is not:

$$Max\ m(t_f) \text{ but } Min \int_{t_0}^{t_f} |u(t)| dt$$

1.2 Single shooting

In order to solve this problem, we first use the single shooting method (based on Pontryagin's Maximum Principle, see [4]), which changes our optimal transfer problem in the resolution of an equation of the form $S(z) = 0$, where S is the shooting function associated to the orbital transfer problem. The main advantage of this method is that there is no requirements on the structure of the control.

We introduce the costate p, and define the Hamiltonian

$$H(t, x, p, u) = |u(t)| + (p(t)|f(t, x, u))$$

which for our problem is of the form:

$$\begin{aligned} H(t, x, p, u) &= (1 - \beta T_{max} p_m(t))|u(t)| \\ &\quad + \tfrac{T_{max}}{m(t)}(B(x(t))u(t)|p(t)) + (a(x(t))|p(t)) \end{aligned}$$

Then the application of Pontryagin's Maximum Principle and optimality necessary conditions gives the expression of the optimal control, or ***H*-minimal command**, which minimizes the Hamiltonian H:

If ${}^tB(x(t))p(t) \neq 0$ then let us define the **switching function** ψ:

$$\psi(t, x, p) = 1 - \beta T_{max} p_m(t) - \tfrac{T_{max}}{m(t)}|{}^tB(x(t))p(t)|$$

Then we have the following expression of the control

$$\begin{cases} u(t) = -\frac{{}^tB(x(t))p(t)}{|{}^tB(x(t))p(t)|} & , if\ \psi(t, x, p) < 0 \\ u(t) = -\alpha\frac{{}^tB(x(t))p(t)}{|{}^tB(x(t))p(t)|} \quad , \alpha \in [0, 1] & if\ \psi(t, x, p) = 0 \\ u(t) = 0 & , if\ \psi(t, x, p) > 0 \end{cases}$$

Else if ${}^tB(x(t))p(t) = 0$ we have

$$\begin{cases} u(t) \in S(0, 1) & \text{if } 1 - \beta T_{max} p_m(t) < 0 \\ u(t) \in B_f(0, 1) & \text{if } 1 - \beta T_{max} p_m(t) = 0 \\ u(t) = 0 & \text{if } 1 - \beta T_{max} p_m(t) > 0 \end{cases}$$

We can see that this control can be discontinuous, as its norm switches between 0 and 1 at zeros of the switching function ψ.

We now make two assumptions, which are numerically verified:

(H1) We assume that ${}^tB(x(t))p(t)$ is non zero for all $t \in [t_0, t_f]$

(H2) There is no **singular arc**, that is to say that we do not have $\psi(t, x, p) = 0$ on an whole interval.

Even under these assumptions, the application of the single shooting can be quite tricky, which is in particular due to the fact that the control structure is not known a priori. To jump over this difficulty, we use homotopic method which will be explained in the next chapter.

Concerning the initial and terminal conditions of the Two Points Boundary Value Problem ($TPBVP$) associated to the shooting function, they will be specified in the third chapter as they depend on MfMax's version. Same remark for the shooting function unknowns.

Chapter 2

Homotopy method

2.1 General idea

The objective is here to find a zero of a function f, which is in our case the shooting function associated to the maximization of the payload. In order to do so we introduce another function r whose zeros are easier to compute than the one of f. We relate those 2 functions by the mean of what we call the homotopic parameter λ. If the function r and its link with f are well chosen, there exist a smooth zero path starting at a zero of r and ending at a zero of f. As one has already guessed, all difficulties lie in the way to choose r and its relation with f but also in the tracking of the zero path. More detailed explanation than the one given in the next section can be found in [6].

2.2 Theoritical preliminaries

Definition(*homotopy*):

An homotopy is an application: $S : \bar{\Omega} \times [0,1] \to \mathbf{R}^n$ with $\Omega \subset \mathbf{R}^n$ open and bounded and S continuous.

Definition(*admissibility*):

An homotopy map h is called *admissible* with respect to 0 and Ω if and only if : $S^{-1}(0) \cap \partial\bar{\Omega} \times [0,1] = \emptyset$

We will now go back to our functions r and f from $\bar{\Omega} \in \mathbf{R}^n$. We define an homotopy map S as follow:

$$S : \begin{array}{ccc} \bar{\Omega} \times [0,1] & \to & \mathbf{R}^n \\ (z,\lambda) & \mapsto & S(z,\lambda) \end{array}$$

Verifying:

$$\begin{array}{rcl} S(z,0) & = & r(z), \forall z \in \bar{\Omega} \\ S(z,1) & = & f(z), \forall z \in \bar{\Omega} \end{array}$$

It is quite clear that S links r to f. Moreover, we have a zero of r which is a zero of $S(.,0)$ and we seek a zero of f which is a zero of $S(.,1)$. The following theorem gives us some information about the existence of zero paths linking those 2 zeros.

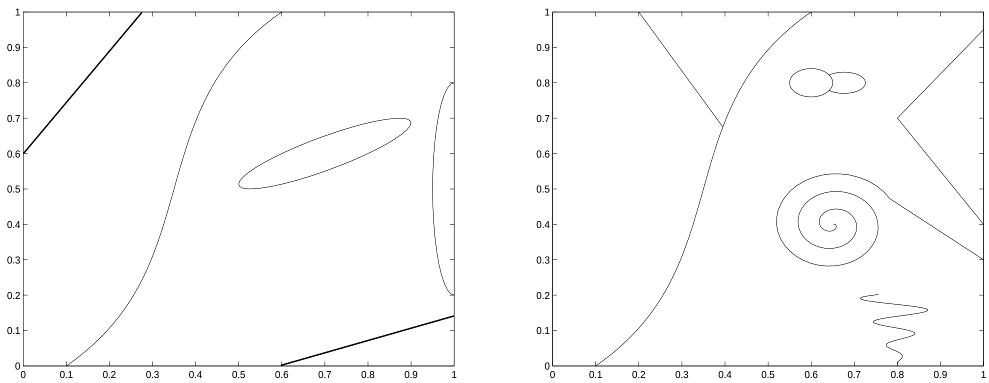
Theorem:

Set Ω an open bounded subset of $\mathbf{R}^n$. Set $S : \bar{\Omega} \times [0,1] \to \mathbf{R}^n$ continuously differentiable and such that :
(a) $\forall (z,\lambda) \in \{(z,\lambda) \in \Omega \times [0,1] \textit{ such that } S(z,\lambda) = 0\}$ the jacobian matrix $S' = [\frac{\partial S}{\partial z_1} ... \frac{\partial S}{\partial z_n} \frac{\partial S}{\partial \lambda}]$ is of full rank n.
(b) $\forall z \in \{z \in \Omega \textit{ tel que } S(z,0) = 0\} \cup \{z \in \Omega \textit{ such that } S(z,1) = 0\}$ the marix $[\frac{\partial S}{\partial z_1} .. \frac{\partial S}{\partial z_n}]$ is of full rank n.

Then $\{(z,\lambda) \in \Omega \times [0,1] \textit{ such that } S(z,\lambda) = 0\}$ is made of :
(i) a finite number of closed curves (of finite length) in $\bar{\Omega} \times [0,1]$.
(ii) a finite number of arcs (of finite length) having their terminal points in $\partial\Omega \times [0,1]$.
Curves (i) and (ii) are seperated and continuously differentiable.

Proof : see [7]

Figure 4 shows some possible and impossible paths.



possible paths impossible paths

Figure 4: Possible and impossible paths

In order to track zero paths, we have different methods. A first one, called discrete continuation, simply consist of solving $S(z,\lambda_i) = 0$ for an increasing sequel $(\lambda_i)_{i=1,..,k}$ with $\lambda_1 = 0$ and $\lambda_k = 1$. Of course, for solving $S(z,\lambda_i) = 0$, we use the known zero z_{i-1} of $S(z,\lambda_{i-1})$.

A second method, called piecewise linear (PL), follows the zero path by building a piecewise linear approximation of h. One can refer to [6] for detailed method and to [5] for an operational code dedicated to our orbital transfer.

A last one, called the predictor corrector (PC) method or differential continuation is the one we use and is explained in the following section.

2.3 Differential homotopy

Let us assume that the considered homotopy $S(z, \lambda)$ is sufficiently regular ($\mathcal{C}^2$) and that the zero path that comes from $(z_0, 0)$ is a differentiable curve C. We can parametrize this curve by the curvilinear abscissa s and suppose we have the relation :

$$\begin{cases} (i)\ |(\frac{\partial z}{\partial s}, \frac{\partial \lambda}{\partial s})| = 1 \\ (ii)\ S(z(s), \lambda(s)) = 0 \\ (iii)\ S'(z(s), \lambda(s)) \text{ if of full rank } n \end{cases}$$

Differentiation of (ii) with respect to s give us :

$$(iv) \left[\frac{\partial S}{\partial z}(z(s), \lambda(s)), \frac{\partial S}{\partial \lambda}(z(s), \lambda(s))\right] \left[\begin{array}{c} \frac{\partial z}{\partial s}(s) \\ \frac{\partial \lambda}{\partial s}(s) \end{array} \right] = 0.$$

(i) and (iv) determine (except for the direction) the unit tangent vector to C.

To determine the direction, we introduce the augmented jacobian matrix:

$$A(s) = \left[\begin{array}{cc} \frac{\partial z}{\partial s}(s) & \frac{\partial \lambda}{\partial s}(s) \\ \frac{\partial S}{\partial z}(z(s), \lambda(s)) & \frac{\partial S}{\partial \lambda}(z(s), \lambda(s)) \end{array} \right]$$

(iii) implies that $A(s)$ is non singular and that :

$$(v) sgn(det(A(s))) = sgn(det(A(0)))$$

By setting the first direction of the tangent vector ($\frac{\partial \lambda}{\partial s}(s) > 0$ for example) we are able to compute the unique unit tangent vector to C with (i), (iv) and (v). We denote $t(S'(z, \lambda))$ the tangent vector to S in (z, λ).

Hence, following the zero path of h is equivalent to the integration of the initial value problem (IVP):

$$(IVP) \begin{cases} (\dot{z}(s), \dot{\lambda}(s)) = t(S'(z(s), \lambda(s))) \\ (z(0), \lambda(0)) = (z_0, 0) \end{cases}$$

Integration of the (IVP)

Here, we describe the method used in L.T Watson's *HOMPACK90* software ([8]).

In order to numerically integrate our (IVP), we have one more information on $(z(s), \lambda(s))$. This information says that $S(z(s), \lambda(s)) = 0$, that's why we can perform our integration as follow (figure 5):

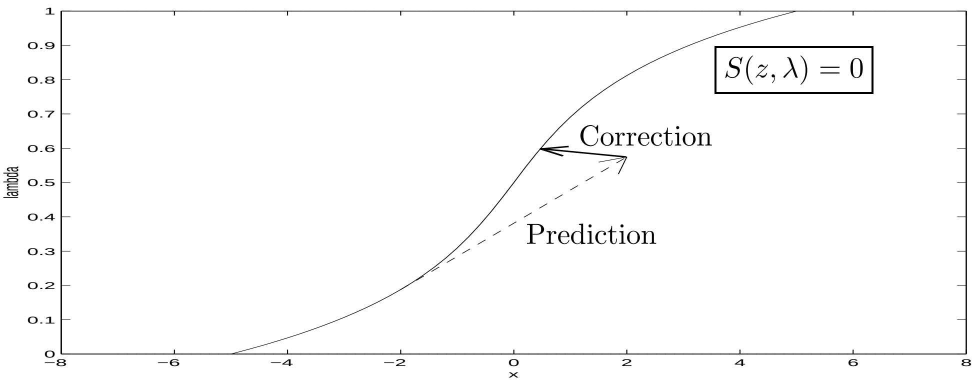


Figure 5: Predictor-Corrector scheme

If we note $v = (z, \lambda)$, we can decompose an integration step in two main phases : prediction and correction.

The prediction step consists in a simple scheme (for instance Euler):

$$u^{n+1} = v^n + d.t(S'(v))(d \text{ is the steplength})$$

The correction phase consists in getting back on the zero path which is not (with some hope) too far :

$$v^{n+1} = \underset{h(\omega)=0}{argmin} \frac{1}{2}|\omega - u^{n+1}|^2$$

TSis correction is performed with Newton steps, which are supposed not to be too expensive as we are not far from the solution.

The main advantage of this method is that the steplength of the prediction can take into account the previous predictions so that if the zero path is regular, the following can be very fast.

But there is also a drawback which take place in the fact that for each prediction and correction step, we have to evaluate the jacobian of the homotopy, which can sometimes be ill conditioned, introducing numerical difficulties. Indeed, as for our problem the homotopy will be a shooting function parametrized with λ, it will be computed by the integration of the (IVP). That's why we must have a good adequation between the integration step error and the step of finite differences which will be used for computing the approximation of the jacobian of the homotopy.

2.4 Pseudo code algorithm

Algorithm
Initialization
Solve the *easy* problem $\rightarrow (z_0, 0)$ such that $S(z_0, 0) = 0$.
Choose **nstepmax** (Maximum number of integration step)
Choose **ARCRE, ARCAE**, accuracies (relative and absolute) for zero path tracking.
Choose **ANSRE, ANSAE**, accuracies (relative and absolute) for the solution in $\lambda = 1$.
$y_0 := (z_0, \lambda_0) := (z_0, 0)$
nstep $:= 0$
Choose **itmax** (maximum number of newton correction used to go back to the zero path).
Body
Begin
Repeat
Compute the unit tangent vector $t(S'(z_{nstep}, \lambda_{nstep}))$
label 10
$u_{nstep+1} := y_{nstep} + d.t(S'((z_{nstep}, \lambda_{nstep})))$ (***Prediction***)
For i = 1 to itmaxdo (***Correction***)
$c := S'(u_{nstep+1})^{+} S(u_{nstep+1})$
$u_{nstep+1} := u_{nstep+1} - c$ (pseudo Newton step)
If $|c|_2 \leq ARCRE|u_{nstep+1}|_2 + ARCAE$
Then
Go to label 20
End If
End For
$d = d/2$ (non convergence, we decrease steplength)
Go to label 10
label 20
nstep := nstep + 1
$y_{nstep} := u_{nstep}$
Optimal steplength estimation
Until $y_{nstep}(1) \geq 1$ ou nstep $\geq$ nstepmax
If nstep $\geq$ nstepmax Then
Not enough integration step allowed
Else
Solve $S(z, 1) = 0$ with y_{nstep} as initialization
End If
End

2.5 Application to our problem

In the first chapter we saw the application of the Pontryagin's maximum principle to our orbital transfer problem.

For the application of the homotopy method, the target function (previously named f) is obviously the shooting function associated to the payload maximization. An easier connected problem is the one of minimization of energy ($min \int_{t_0}^{t_f} |u(t)|^2 dt$) which is smoother due to the criterion. We choose to link those 2 problems by the introduction of the homotopic parameter (λ) in the criterion as follow:

$$\int_{t_0}^{t_f} \lambda|u(t)| + (1-\lambda)|u(t)|^2 \ dt$$

It is quite clear that for $\lambda = 0$ we minimize the energy and that for $\lambda = 1$ we minimize the consumption. We already applied the Pontryagin's principle to the homotopy when $\lambda = 1$, the only difference for $\lambda < 1$ is the computation of the optimal control u which becomes continuous (that is why the problem is easier to solve).

Taking the first chapter notations, the hamiltonian can be written as follow:

$$\begin{array}{rcl} H(t,x,p,u) & = & (1-\lambda)|u(t)|^2 + (\lambda - T_{max}\beta p_m(t))|u(t)| \\ & & + \frac{T_{max}}{m(t)}(B(x(t))u(t)|p(t)) + (a(x(t))|p(t)) \end{array}$$

We repeat the assumption **(H1)** of the first chapter (${}^tB(x(t))p(t)$ is non zero $\forall t \in [t_0, t_f]$). We then have the expression of the optimal control which minimizes H (for $\lambda < 1$):

We define the unconstraint modulus function $\alpha(.)$:

$$\alpha(t,x,p) = \frac{T_{max}\beta p_m(t) + \frac{T_{max}}{m(t)}|{}^tB(x(t))p(t)| - \lambda}{2(1-\lambda)}$$

Then we have the following expression of the control:

$$\left\{ \begin{array}{rcll} u(t) & = & 0 & ,if \ \alpha(t,x,p) < 0 \\ u(t) & = & -\alpha(t,x,p)\frac{{}^tB(x(t))p(t)}{|{}^tB(x(t))p(t)|} & ,if \ \alpha(t,x,p) \in [0,1] \\ u(t) & = & -\frac{{}^tB(x(t))p(t)}{|{}^tB(x(t))p(t)|} & ,if \ \alpha(t,x,p) > 1 \end{array} \right.$$

Now, we can dive in some of the specificities of the 2 different implementation and expression of the orbital transfer problem. Namely **MfMax-v0** and **Mfmax-v1**.

Chapter 3

Specificities of MfMax

3.1 Common particularity

3.1.1 Initial condition homotopy

The initialization of the homotopy linking energy to mass needs the resolution of the energy problem. Though it is easier to solve than the mass problem, it remains quite hard for low thrust (less than 5 Newton).

In order to initialize the homotopy, we introduce another one which is based on this observation: **if initial and final orbits are equal, the optimal thrust strategy consist is no thrust at all**. This null strategy corresponds to null adjoint state (ie, null shooting function unknowns). We then have a trivial problem which can be linked to the energy problem by the mean of the $S_{orbit}(z, \lambda)$ homotopy map associated to the following TPBVP:

$$
\begin{cases}
min \int_0^{t_f} |u(t)|^2 dt \\
\text{State and costate dynamics of the transfer problem} \\
t^f \text{fixed} \\
P(0) = (11625\lambda + (1-\lambda)42165)\ km \quad , \quad P(t_f) = 42165\ km \\
e_x(0) = 0.75\lambda \quad , \quad e_x(t_f) = 0.0 \\
h_x(0) = 0.0612\lambda \quad , \quad h_x(t_f) = 0.0 \\
\text{Other boundary conditions}
\end{cases}
$$

For $\lambda = 1$, S_{orbit} is the problem of minimization of energy, and in $\lambda = 0$ we know the solution. It is quite easy to go from $\lambda = 0$ to $\lambda = 1$, simple discrete continuation is enough.

This discrete continuation allows us to find initializations for thrust downto 0.1 Newton.

3.2 MfMax-v0

3.2.1 Fixed parameters

In this version, ***we fix the final time*** t_f. If we solve the problem and draw the evolution of the final mass with respect to the final time, this is what we get:

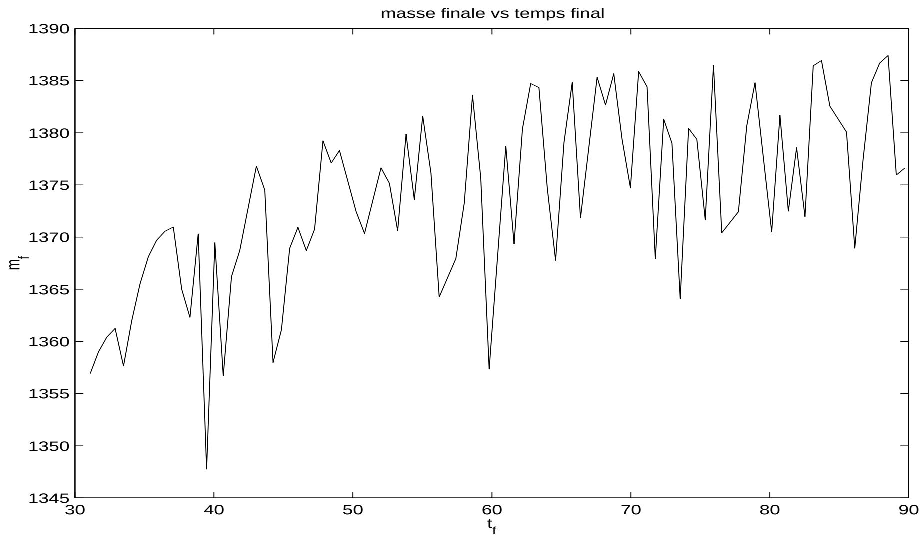


Figure 6: Final mass with respect to final time

We can see that the final mass we find is not increasing. This indicates that our resolution leads sometime to local solutions. The most annoying is that the loss of mass could be very important.

Those local solutions are probably related to the periodicity of the problem with respect to the longitude. That is why we fix the last free parameter: L^f. We thus set a rendez-vous on the final orbit but more important, the number of revolutions of the transfer.

To set this final longitude (L^f) we proceed as for t_f. We first solve the minimum longitude transfer problem which gives us the minimum transfer longitude L^f_{min}. We then apply a multipliar coefficient c_{Lf} as follow:

$$L^f = c_{Lf}(L^f_{min} - L^0) + L^0$$

As the minimum time problem and the minimum longitude problem are quite similar, it is not surprising that like ${t^f}_{min}$ we have an empiric law as follow:

$$(L^{T_{max}}_{f_{min}} - L^0)T_{max} \approx C^T = L^{ref}_f$$

More precisely, with our orbital transfer we have:

$$L_{ref}^f \approx 267.54rad.$$

Note that $\frac{L_{ref}^f}{2\pi}$ is the minimum number of revolutions for a thrust of 1 Newton:

$$L_{ref}^f = L_{f_{min}}^{1N.} - L^0$$

From now on, the transfer problem will be defined by the 2 multiplier coefficient c_{tf} and c_{Lf}.

Note: fixing L^f introduce a slight modification on the initial condition homotopy which does not accept 0 as solution at $\lambda = 0$ anymore. Yet, for reasonable L^f, the h_{orbit} map still have easy zero.

3.2.2 Optimal final longitude

One can wonder if there is a good c_{Lf} to choose for a given c_{tf} and T_{max}. To answer to this question the first thing to do is to look at the evolution of the final mass with respect to c_{Lf}:

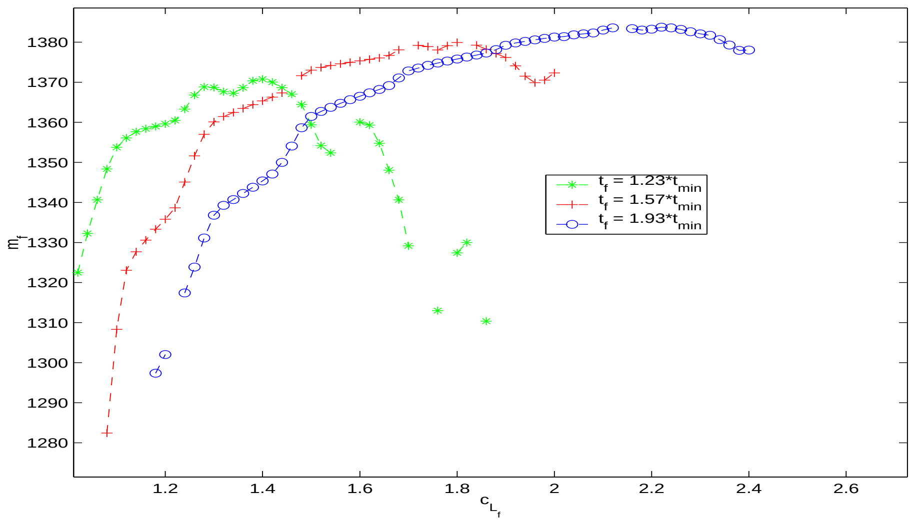


Figure 7: m_f (kg.) vs. c_{Lf} for 10N. and various c_{tf}

On this figure one can see that for a given c_{tf} there is a c_{Lf} which maximize the final mass. This optimal c_{Lf} will be called $c_{L^f_{opt}}$. For the moment the research of $c_{L^f_{opt}}(T_{max}, c_{tf})$ requires the resolution of many final mass problem which could be very expensive (in term of execution time). Moreover, figure 7 also shows local maximum which means that one should have a good approximation before searching $c_{L^f_{opt}}(T_{max}, c_{tf})$.

In order to remedy to those drawbacks one could wish to find $c_{L^f_{opt}}(T_{max}, c_{tf})$ at a cheapest price. The next figure shows the evolution of the final mass with respect to c_{Lf} for the maximum payload criterion but also for the energy one (this is the resulting final mass when minimizing the energy):

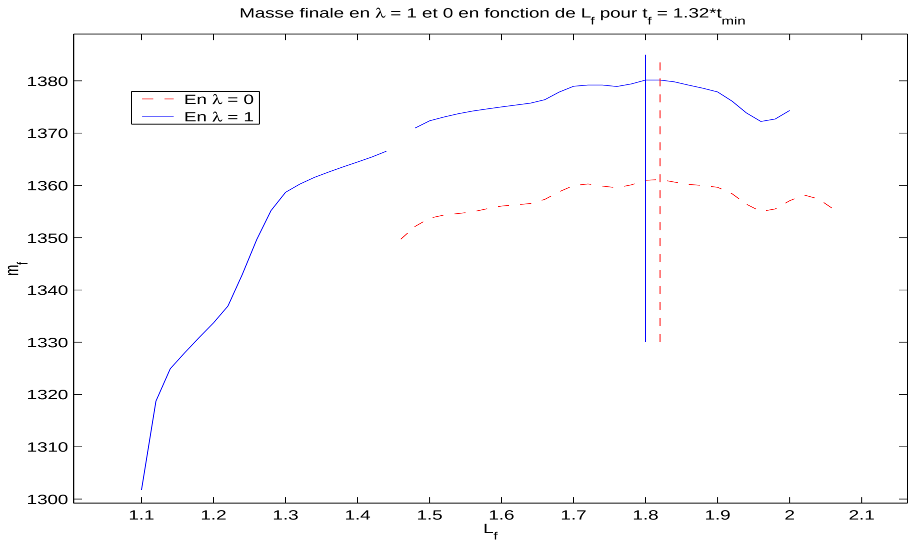


Figure 8: m_f (kg.) vs. c_{Lf} for 10N. and the two criteria

In this figure (8) one can see that for the 2 criteria the maximum final mass happens for the same c_{Lf} (the difference is the testing's step). We can then find $c_{L^f_{opt}}(T_{max}, c_{tf})$ by resolving energy problem which is much faster than the previous research.

Note: by analyzing the evolution of energy consumption with respect to c_{Lf} we can conclude that the optimal c_{Lf} (the one which minimizes energy) has nothing to do with $c_{L^f_{opt}}(T_{max}, c_{tf})$. Moreover, as the differential continuation from energy to final mass criterion just slightly modified L^f (in case it is free), we can conclude that even if we found the global solution of the energy problem, it would not have led us to the global solution of the final mass problem (always for free L^f). This legitimate our choice of fixing L^f.

With this strategy for searching $c_{L^f_{opt}}(T_{max}, c_{tf})$ we were able to draw the evolution of $c_{L^f_{opt}}(T_{max}, c_{tf})$ with respect to c_{tf} and for various T_{max}:

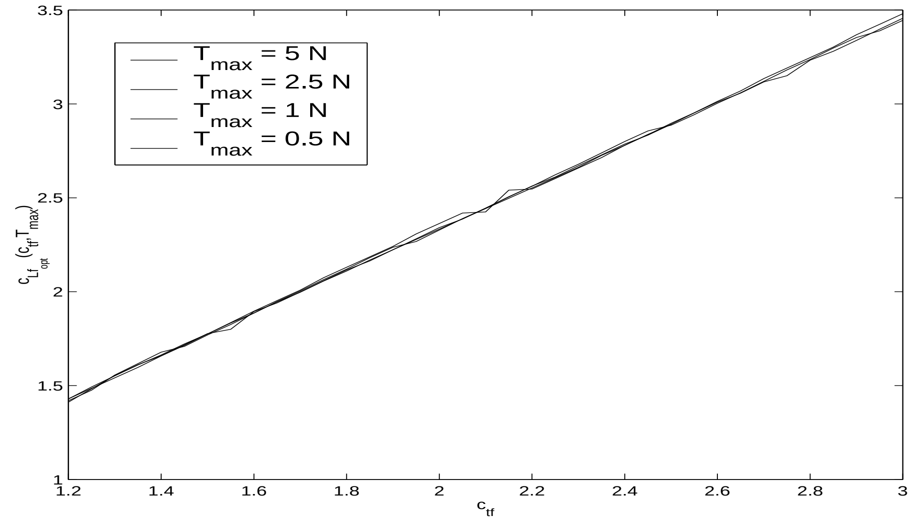


Figure 9: $c_{L^f_{opt}}$ *vs.* c_{tf} *for various* T_{max}

On this figure, it is quite obvious that $c_{L^f_{opt}}$ nearly do not depends on c_{tf} or T_{max}. This is very usefull to approximate $c_{L_f^{opt}}$ either to initialize a research of it or only to solve the final mass problem with a good final mass not too far from the best one.

The next figure shows the evolution of the final mass with respect to c_{tf} for all previously described resolution strategies:

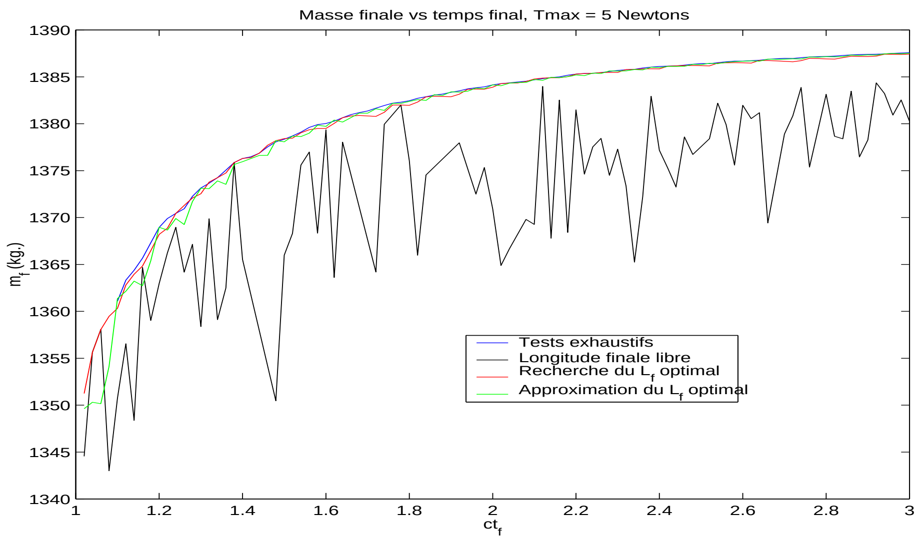


Figure 10: m_f *(kg.) vs.* c_{tf} *for different resolution strategies*

We can see that the approximation of $c_{L^f_{opt}}$ gives us good results and that the free L^f strategy is really not the good one.

3.2.3 Transversality condition

In this section, we will quickly explicit the shooting unknowns and the initial and terminal conditions.

Let us recall the boundary condition expressed in the first chapter:

$$\begin{array}{lllllll} P^0 & = & 11625km & P^f & = & 42165km \\ e_x^0 & = & 0.75 & e_x^f & = & 0 \\ e_y^0 & = & 0 & e_y^f & = & 0 \\ h_x^0 & = & 0.0612 & h_x^f & = & 0 \\ h_y^0 & = & 0 & h_y^f & = & 0 \\ L^0 & = & \pi & L^f & = & \text{some multiplier of minimum longitude} \\ m^0 & = & 1500kg & m^f & free & \end{array}$$

From this we see that the shooting function unknown z is:

$$z = (p_P(0), p_{e_x}(0), p_{e_y}(0), p_{h_x}(0), p_{h_y}(0), p_L(0), p_m(0))$$

As m_f is free, we add to the terminal condition:

$$p_m(t_f) = 0$$

We then have all needed boundary condition to define the shooting function.

3.2.4 Additionnal criterion

In this version of MfMax, there is a second choice of criterion linking energy to mass criterion. We call it the power criterion, and it is as follow:

$$Min \int_{t_0}^{t_f} |u(t)|^{(2-\lambda)} dt$$

We clearly have the desired criterion in $\lambda = 0$ and $\lambda = 1$. This power criterion leads us to a different optimal control computation which for $\lambda < 1$ becomes as follow:

We repeat the assumption **(H1)** of the first chapter ($B^T(x(t))p(t)$ is non zero $\forall t \in [t_0, t_f]$). We then have the expression of the optimal control which minimizes H:

We define the unconstraint modulus function $\alpha(.)$:

$$\alpha(t,x,p) = \begin{cases} 0 \text{ if } T_{max}\beta p_m(t) + \frac{T_{max}}{m(t)}|B^T(x(t))p(t)| < 0 \\ \left(\frac{T_{max}\beta p_m(t) + \frac{T_{max}}{m(t)}|B^T(x(t))p(t)|}{2(1-\lambda)}\right)^{\frac{1}{1-\lambda}} \text{ if not} \end{cases}$$

Then we have the following expression of the control:

$$\begin{cases} u(t) &= 0 & ,if\ \alpha(t,x,p) < 0 \\ u(t) &= -\alpha(t,x,p)\frac{B^T(x(t))p(t)}{|B^T(x(t))p(t)|} & ,if\ \alpha(t,x,p) \in [0,1] \\ u(t) &= -\frac{B^T(x(t))p(t)}{|B^T(x(t))p(t)|} & ,if\ \alpha(t,x,p) > 1 \end{cases}$$

3.3 MfMax-v1

3.3.1 Specific formulation

The first difference with the **v0** version is that t_f **is free** and L^f **is fixed**. To fix L^f we use the same way as in MfMax-v0. The problem is then caracterized by the multipliar coefficient c_{L^f} which is applied to L^f_{min}.

Moreover, as the transfer is smoother when integrating with respect to the longitude, we take L as integrating variable. The drawback is that we cannot simply applied the maximum principle to the system if it is expressed with respect to L instead of the time t (autonomous problem). In fact the dynamics remains approximatively similar but the hamiltonian is no more linear in the control. That implies really complex optimal control which are root of a non trivial fourth degree polynomial and then not implementable.

To bypass this difficulty, we start with the autonumous system. As t^f is free, we first have to rewrite the orbital transfer system in order to have fixed integration bound. To do so we set:

$$t(s) = (t_f - t_0)s + t_0,\ s \in [0,1]$$

The rewrite problem can be expressed as follow:

$$(P_\lambda) \begin{cases} min & \int_0^1 (1-\lambda)|u(s)|^2 + \lambda|u(s)|ds,\ \ s \in [0,1] \\ \dot{x}(s) &= t_f * f(s,x,u) \\ \dot{t}_f(s) &= 0 \\ t_f(0),\ \ t_f(1) & \text{free} \\ |u(s)| &\leq 1 \end{cases}$$

We can then add t_f to the state:

$$\tilde{x} = (P, e_x, e_y, h_x, h_y, L, m, t_f)$$

, set:

$$\tilde{f}_i(s,\tilde{x},u) = t_f * f_i(s,x,u), i = 1,..7$$

, and:

$$\tilde{f}_8(s,x,u) = 0$$

The problem is then:

$$(P_\lambda) \begin{cases} min & \int_0^1 (1-\lambda)|u(s)|^2 + \lambda|u(s)|ds,\ \ s \in [0,1] \\ \dot{\tilde{x}}(s) &= \tilde{f}(s,x,u) \\ |u(s)| &\leq 1 \end{cases}$$

With boundary condition:

$$\left\{\begin{array}{llllllll} & \text{IC} & & & & \text{TC} & \\ \tilde{x}_1(0) & = & P^0 & , & \tilde{x}_1(1) & = & P^f \\ \tilde{x}_2(0) & = & e_x^0 & , & \tilde{x}_2(1) & = & e_x^f \\ \tilde{x}_3(0) & = & e_y^0 & , & \tilde{x}_3(1) & = & e_y^f \\ \tilde{x}_4(0) & = & h_x^0 & , & \tilde{x}_4(1) & = & h_x^f \\ \tilde{x}_5(0) & = & h_y^0 & , & \tilde{x}_5(1) & = & h_y^f \\ \tilde{x}_6(0) & = & L^0 & , & \tilde{x}_6(1) & = & L^0 + c_{L^f}(L^f_{min} - L^0) \\ \tilde{x}_7(0) & = & m^0 & , & \tilde{x}_7(0) & & \text{free} \\ \tilde{x}_8(0) & & \text{free} & , & \tilde{x}_8(0) & & \text{free} \end{array}\right\}$$

Applying the Pontryagin's maximum principle, the shooting function unknown becomes:

$$z = (\tilde{x}_8(0), p_P(0), p_{e_x}(0), p_{e_y}(0), p_{h_x}(0), p_{h_y}(0), p_L(0), p_m(0))$$

Transversality condition gives us additionnal boundary conditions which are:

$$\left\{\begin{array}{lllll} p_{\tilde{x}_7}(1) & = & p_m(1) & = & 0 \\ p_{\tilde{x}_8}(0) & = & p_{t_f}(0) & = & 0 \\ p_{\tilde{x}_8}(1) & = & p_{t_f}(1) & = & 0 \end{array}\right.$$

H-minimum command computation is the same as the one desribed in the homotopic method application. The $(TPBVP)$ associated to the autonomous system is:

$$(TPBVP_t)\left\{\begin{array}{l} \left.\begin{array}{lll} \dot{\tilde{x}}(s) & = & \tilde{f}(s,\tilde{x},p,u) \\ \dot{p}(s) & = & -\frac{\partial H(s,\tilde{x},p,u)}{\partial \tilde{x}} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{lll} y & = & (\tilde{x},p) \\ \dot{y}(s) & = & \varphi(s,y,u) \end{array}\right. \\ \text{Optimal control } u \\ \text{Boundary condition} \end{array}\right.$$

We then apply a change of variable in order to integrate the $TPBVP$ with respect to L (y_6). This change of variable is possible because the longitude is strictly increasing wiht the time ($\dot{y}_6(s) > 0$). We then set:

$$\tilde{y} = (y_1, ..., y_5, s, y_7, ..., y_16)$$

And the $TPBVP$ is the following one:

$$
(TPBVP)\left\{\begin{array}{l}
\frac{\partial \tilde{y}}{\partial y_6} = \frac{1}{\varphi_6(s,y,u)}\dot{\tilde{y}} = \tilde{\varphi}(y_6, \tilde{y}, u) \\
\text{optimal control } u \\
\text{(Boundary condition)}\left\{\begin{array}{lcl}
\tilde{y}_1(0) & = & P^0 \\
\tilde{y}_2(0) & = & e_x^0 \\
\tilde{y}_3(0) & = & e_y^0 \\
\tilde{y}_4(0) & = & h_x^0 \\
\tilde{y}_5(0) & = & h_y^0 \\
\tilde{y}_6(0) & = & 0 \\
\tilde{y}_7(0) & = & m^0 \\
\tilde{y}_{16}(0) & = & 0 \\
\tilde{y}_1(L^f) & = & P^f \\
\tilde{y}_2(L^f) & = & e_x^f \\
\tilde{y}_3(L^f) & = & e_y^f \\
\tilde{y}_4(L^f) & = & h_x^f \\
\tilde{y}_5(L^f) & = & h_y^f \\
\tilde{y}_6(L^f) & = & 1 \\
\tilde{y}_{15}(L^f) & = & 0 \\
\tilde{y}_{16}(L^f) & = & 0
\end{array}\right.
\end{array}\right.
$$

This $TPBVP$ is the one which is used in **MfMax-v1**.

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